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13. ABSTRACT (Maximum 200 words)

The proposer has investigated the dynamics of externally forced, weakly coupled, non-linear oscillators. The objective in this project is to use the constructs of group theory and equivalent bifurcation analysis to produce an in-depth picture of the dynamics of such oscillators. In this report, he reviews the concepts of group theory and equivalent bifurcation theory, derive the mathematical model of a three element oscillator, classify the possible solutions of the system, and discuss the predicted dynamics of the weakly coupled cyclic system. Specifically, the proposer derives the equations of motion for weakly non-linear dynamics and applied the method of averaging to reduce the system of non-linear coupled differential equations to an autonomous dynamical system. The averaged equations of motion prove to be a low order polynomial normal form for the general case of cyclically coupled and harmonically excited mechanical systems. An isotropy sub-group lattice is used to obtain information on fixed point solutions of the averaged system of equations, which correspond to periodic motions in the original system. The author shows that, in the case of the free vibration model, there can be three possibilities of solution for each oscillator depending upon initial conditions.

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STRUCTURAL DYNAMICS OF NONLINEAR MECHANICAL SYSTEMS WITH CYCLIC SYMMETRY

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U.S. Army Research Office

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When the coupling in the individual substructures is weak (of the same order as the nonlinearity), all the modes of vibration of the structure have nearly identical frequencies, and the response of the system is expected to be quite complex. The analysis of nonlinear response in the weakly nonlinear case is addressed in the thesis of the ASSERT student, and the framework for analysis is described in this report. Chapter 1 gives a brief background to the literature. Chapter 2 presents a brief introduction to the "equivariant bifurcation theory". In chapter 3, the essential models are developed. They include a discussion of the forced as well as parametrically excited and weakly nonlinear cyclic structures. The asymptotic method of averaging us used to derive a set of amplitude equations whose solutions give the steady state periodic solutions of the cyclic structure. Before the amplitude equations can be solved, it is more important to identify the various symmetry classes of solutions possible, and this is accomplished through the use of "Isotropy subgroup lattice" ideas detailed in chapter 4. The final chapter 5 outlines the remaining steps of analysis which are now being completed by the student while writing the doctoral dissertation.

As a part of the training of the student in "equivariant bifurcation theory", the problem of flow-induced oscillations in a cantilever tube conveying a pulsatile flow was investigated. Interesting results for spatial motions of the cantilever tube, modeled as a space elastica, were obtained. They are included in two works which form the appendices A and B of this report.

Publications:

One conference paper and journal paper (submitted) resulted from the initial study undertaken by the ASSERT student. The conference paper was published in a special issue of the journal ZAMM (included in Appendix A), and the journal paper is in review (abstract included in Appendix B).

Personnel:

Two faculty members and one graduate student at Purdue University, and one faculty member at Southern University, were partially funded under parent contract. Only a graduate student was funded through the ASSERT grant.

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Mr. Chris Folley is writing his doctoral thesis "Nonlinear Dynamics of N Identical Weakly Coupled Forced Mechanical Oscillators", and is expected to finish by May 1997.

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CHAPTER 1
INTRODUCTION

Weakly coupled oscillators have played a large role in the fields of physics, mathematics, biology and engineering for some time with applications in areas ranging from the dynamics of large space antennas (Levine-West and Salama [1]) to neural pattern generation (Ermentrout and Kopell [2]). The formulation of such a wide array of problems could not possibly be identical, though the resulting dynamics tend to be similar. The interest here is in a ring of weakly coupled nonlinear oscillators with identical external forcing applied to each, with the weak coupling generated by linear extensional, elastic elements. The engineering application of interest is in the vibrations of bladed disk assemblies which is of considerable concern to the structural vibration design of axial compressors, particularly when designing against the phenomenon of rogue blade failure in such turbomachines. This failure results in a small number of contiguous blades attaining excessive amplitudes of vibration, and failing during certain engine operation conditions, while the remaining blades vibrate with small amplitudes and are well within fatigue and wear limits. We use a generic model for each oscillator to keep the discussion as general as possible, and to point out differences between linear and nonlinear oscillators. The external forcing on the oscillators will be chosen to be, effectively, a parameter that can be varied, but with the property that the amplitude applied to each blade is equal while the phase value may vary from oscillator to oscillator in a prescribed fashion. We stress that we are considering weak coupling of identical oscillators with external periodic forcing since it has been shown (Vakakis [3], Samaranayake et al. [4]) that all oscillators are in resonance when weakly coupled, as opposed to the strong coupling case where only two system modes are in resonance.

The primary motivation for this discussion comes from Samaranayake et al. [5] where the model we analyze in this work is presented, but the analysis there is restricted to the three oscillator case with only one oscillator being forced. Not many studies have been published dealing with the dynamics of externally forced, weakly coupled, nonlinear oscillators, though many studies on unforced nonlinear vibrations, and forced response of strongly coupled linear oscillators have been performed. In recent years, Vakakis [3] and King and Vakakis [6] have studied free and forced response of weakly coupled nonlinear oscillators with an eye towards bladed disk assemblies. These authors, along with Vakakis and Caughey [7], have successfully attempted to show the connection between mode localization in such systems and the concept of similar and nonsimilar nonlinear normal modes that was introduced by Rosenberg [8] for the vibratory response of nonlinear multi-degree-of-freedom systems. In all these studies, however, only a few equilibrium solutions are shown along with some stability analysis, and therefore form an incomplete illustration of the dynamics. Our objective here is to use the constructs of group theory and equivariant bifurcation analysis to show a far more complete picture of the dynamics that are possible.

Equivariant bifurcation theory is not a new development in mathematics, although its use in the analysis of physical problems is relatively recent. The utility of equivariant bifurcation theory has increasingly been noticed by researchers in many fields in the past 15 years. The seminal publications in this area are the works of Golubitsky and Schaeffer [9], Golubitsky et al. [10], and Gaeta [11] where the foundations for the field are described in terms of the algebraic [9, 10] and geometric [11] properties. Coupled oscillator systems have been analyzed using this kind of analysis in Golubitsky and Stewart [12] and Collins and Stewart [13] for example, but only the work of Ashwin and Swift [14], to the authors' knowledge, has attempted to describe the dynamics of an arbitrary number of weakly coupled oscillators in a systematic way. The interest in that work stems from some neuroscience and biological applications, and, therefore, their network is quite different from ours. Thus, their conclusions regarding dynamics are not applicable, though some of their analysis approach is quite useful. This work [14] in conjunction with [5] form the primary motivation for the research proposal presented here, and with the construct presented in these studies, we will attempt to describe the dynamics of the system of interest.

We proceed in the following fashion. First, a brief introduction to group theory concepts and equivariant bifurcation theory is provided by presenting a set of definitions that describe the terminology and concepts used in the subsequent analysis. We then attempt to show that this analysis, for our purposes, is a generalization of the familiar linear transformation theory of linear algebra. In chapter 3, we present the derivation of the model we wish to use for the physical system with the work of Samaranayake et al. [5] forming the basis of the motivation. Chapter 4 is devoted to a preliminary investigation into classifying the possible solutions for the system. Finally, in chapter 5, we outline the specific work on this problem that will result in a much more complete picture of the dynamics of the weakly coupled cyclic system.

CHAPTER 2

A BRIEF INTRODUCTION TO EQUIVARIANT BIFURCATION THEORY

The field of equivariant bifurcation theory is grounded in the principles of group theory, group representation theory, abstract algebra, and bifurcation theory. The purpose of this section is to place the pertinent concepts in these areas, for our discussion, on a foundation that is appropriate to the objectives of this work. Our interest here is from the point of view of matrix groups, particularly the group of invertible $n \times n$ matrices over the field of real numbers, known as the general linear group and denoted by GL(n). Our approach to equivariant bifurcation theory is to generalize the notions of linear transformations acting on a vector space, and to determine the resulting dynamics for systems that obey certain properties under particular classes, or subgroups, of GL(n). Therefore, section 1 starts by presenting a review of some familiar concepts in linear algebra, which is then followed by some elementary notions of group theory. Necessarily, then, this section is predominately comprised of definitions. Section 2 applies these concepts to dynamical systems in a general framework. The main results of the equivariant bifurcation theory are then stated in the form of theorems 1 and 2.

2.1 Linear Algebraic and Group Theoretic Preliminaries

We begin with a review of some linear algebra that is based predominately on the discussion in Friedberg et al. [15].

Definition 1: The *Cartesian product* of two non-empty sets S and T is the ordered set:

$$S \times T = \{(a,b) \mid \text{ a an element of S, b an element of T}\} .$$

Let S, T and R be non-empty sets. A *binary operation* is a mapping of $S \times T$ into R, i.e., if s is in S, t is in T, and ν is a binary operation, then $\nu(s,t)$ is an element of R.

Definition 2: A *field* F is a set in which two operations called addition and multiplication, denoted + and $\cdot$ respectively, are defined so that for each pair of elements a, b in F, there are unique elements a+b and $a \cdot b$ in F, such that the following conditions hold for all elements a, b, c in F:

F1. $a+b = b+a$ and $a \cdot b = b \cdot a$ (*Commutativity of addition and multiplication*);

F2. $(a+b)+c = a+(b+c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (*Associativity of addition and multiplication*);

F3. There exist unique elements denoted 0 and 1 in F such that $0+a = a$ and $1 \cdot a = a$ (*Existence of identity elements for addition and multiplication*);

F4. For each element a in F and each nonzero element b in F, there exist elements c and d in F such that $a+c = 0$ and $b \cdot d = 1$ (*Existence of inverses for addition and multiplication*); and

F5. $a \cdot (b + c) = a \cdot b + a \cdot c$ (*Distributivity of multiplication over addition*).

As a point of terminology, we will call elements from a field *scalars.*

The two fields of interest here are the real and complex number fields with the standard definitions of multiplication and addition. The definition above is only given for completeness since for the definition of a vector space, the notion of a field is needed.

Definition 3: A *vector space* V over a field F consists of a set of ordered n-tuples of scalars, and two operations called 'addition' and 'scalar multiplication' that are defined as follows. Addition: for each pair of elements x, y in V, there is a unique element x+y in V. Scalar multiplication: for each element a in F, and each element x in V, there is a unique element ax in V. These two properties are called *closure* under addition and scalar multiplication, respectively, and in addition to these two properties, a vector space must also satisfy the following addition and multiplication rules:

Addition:

A1. For each triple x, y, and z in V, $(x + y) + z = x + (y + z)$ (*Associativity of addition*);

A2. There exists an element in V denoted 0 such that $x + 0 = x$ for each element x in V (*Additive identity*);

A3. For each element x in V, there exists an element y in V such that $x + y = 0$ (*Additive inverse*);

A4. For each pair of elements x, y in V, x + y = y + x (*Commutativity of addition*);

Multiplication:

M1. For each pair of scalars a and b in the field F, and each element x inV, (ab)x = a(bx) (*Associativity of scalar multiplication*);

M2. For each element x in V, there exists an element denoted 1 such that 1x = x (*Multiplicative identity*);

M3. For each element a in the field F, and for each pair of elements x and y in V, a(x + y) = ax + ay (*Distributive property* number 1); and

M4. For each pair of elements a and b in the field F, and for each element x in V, (a + b)x = ax + bx (*Distributive property* number 2).

We will refer to elements of the vector space V as *vectors*.

<u>Remarks</u>:

(1). The binary operation of addition in definitions 2 and 3, although denoted similarly, can be quite different. When not clear by the context which is meant, it will be stated.

(2). With the above definition, if $\mathbb{C}$ denotes the field of complex numbers, then the notation $\mathbb{C}^n$ is used to represent the Cartesian product of n complex number fields,

$$\mathbb{C}^n = \underbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}_{\text{n times}}.$$

It is a well-known theorem in linear algebra that a Cartesian product of a finite number of identical fields forms a vector space, and therefore an element of the set $\mathbb{C}^n$ is a *vector* containing an ordered n-tuple of complex-valued scalars. Note that for convenience, the field can be changed from real to complex by making the standard identification z = x + iy, $i = \sqrt{-1}$, which essentially defines an equivalence between the real plane $(\mathbb{R}^2)$ and the complex numbers $(\mathbb{C})$, and results in a more compact notation. We will not exploit any of the geometrical structure inherent in $\mathbb{C}^n$ in our analysis since this identification is simply a convenience of notation, and it reduces the dimensions of the matrices in GL(n) by a factor of four. The field and vector space dealt with in particular for our purposes will be $\mathbb{R}$ and $\mathbb{R}^n$, and $\mathbb{C}$ and $\mathbb{C}^n$, respectively, by this identification;

(3). Note that we have not specified any of the binary operations, either for the field or the vector space, but simply stated their existence and properties. This has been done purposely to keep in mind that we are generalizing the concept of linear transformations in the discussion below.

Definition 4: A *subspace* W of a vector space V is a subset of V that is a vector space under the conditions and properties of definition 3. Let S be a non-empty subset of V (not necessarily a subspace). Then, a vector x in V is said to be a *linear combination* of elements of S if there exist a finite number of elements $y_1, y_2, .., y_m$ in S and scalars $a_1, a_2, .., a_m$ such that $x = a_1y_1 + a_2y_2 + \ldots + a_my_m$. If a subset W of a vector space V consists of all linear combinations of a finite set of vectors $T = \{y_1, y_2, .., y_k\}$, then this set of vectors is said to be the *span* of W, or *span(T) = W.*

Remark: It can be shown by a direct application of definition 3 that span(T) is a subspace of the original vector space V (see, for example, [15, pp. 29-30]).

Definition 5a: A finite set of vectors $y_1, y_2, .., y_k$, belonging to a vector space V, is said to be *linearly dependent* if there exists a finite number of distinct scalars $a_1, a_2, .., a_k$ not all zero, such that $a_1y_1 + a_2y_2 + \ldots + a_ky_k = 0$. If no such set of scalars exists, then S is said to be *linearly independent.*

Definition 5b: A spanning set of linearly independent vectors is called a *basis.* The *dimension* of the subspace spanned by such a basis set of vectors is given by the order of the minimal basis set. It is easily shown that a minimal spanning set for a given subspace forms a basis for that subspace.

Definition 6: A *linear transformation* $f : V \rightarrow V$ is a map of a vector space into itself such that the addition and scalar multiplication operations of the vector space are preserved by the following rules: $f(x_1 + x_2) = f(x_1) + f(x_2)$, and $f(cx) = cf(x)$ where x, x_1, x_2 are elements of V and c is a scalar.

Remarks:

(1). Note that the operations cx and cf(x) are both scalar multiplication since x and f(x) are elements of the same vector space.

(2). This definition opens up an entire chapter in linear algebra involving eigenvalues, eigenvectors, eigensubspaces, etc. that are important concepts we assume the reader is familiar with.

We now abstract these concepts by introducing groups, and thus the above definition forms our point of departure from linear algebra to abstract algebra. The definition of a group that we present is both abstract, and, for our purposes, functional. In the following, X is a non-empty set.

Defintion 7: A *group* is a pair (X, μ) such that μ is an associative binary operation on the non-empty set X that contains an identity element, and every element in X has an inverse contained in X. A *subgroup* H of a group X is a subset of X with the following properties:

(a). If a and b are elements of H, then $\mu(a,b)$ is an element of H, where $\mu(\cdot,\cdot)$ denotes the binary operation (*Closure*);

(b). The subset H contains the identity element (*Existence of Identity*); and

(c). If a is an element of H, then the inverse of a is an element of H (*Existence of Inverses*).

Thus, the subgroup inherits the binary operation defined for the original group from which it is formed.

Remark: The concept of a group, for our purposes, can be abstracted from the definition of a vector space as given in definition 3, by stripping away all but a fundamental set of properties. First, we remove the notion of linear combinations. Next, notice that scalar multiplication is dependent on two different classes of mathematical objects, scalars and vectors, in order to perform the combination. In addition, we eliminate scalar multiplication and leave the operation of addition intact which involves only vectors. Thus, if we remove all properties in definition 3 except for (A1 - A3), we have a group composed of the set V and the binary operation of vector addition. If we include property A4, the group is said to be Abelian since it is a commutative group.

We have given an abstract definition of a group since our discussion of groups in chapter 4 will be both in terms of matrix groups and abstract groups. In this light, suppose we wish to work with two different groups. What mapping will allow us to move back and forth between these groups such that their structure is preserved?

Definition 8: Let $(G,\circ)$ and $(H,*)$ be two groups with associated binary operations. A *homomorphism* of the group $(G,\circ)$ into the group $(H,*)$ is a map T of G into H such that if x and y are a pair of elements in G then $T(x \circ y) = (T(x))*(T(y))$. An *isomorphism* is a one-to-one homomorphism.

A homomorphism, therefore, is a map that preserves the binary operations of the two groups.

As mentioned above, our interest is in the general linear group, GL(n). This group is part of a larger class of groups known as Lie groups, and the results stated below hold for these more general objects. We will not rely on Lie group theory however, and we simply note that we will use this term to denote subgroups of GL(n) with certain technical restrictions that are not pertinent to our discussion. Further note that the term compact Lie group will refer to elements of GL(n) with finite entries. From this point onward, the discussion in this section is based predominately on reference [10].

Definition 9: Let Γ be a Lie group and V a finite-dimensional real vector space. Γ acts linearly on V if there is a continuous mapping, called the *action*, which maps $\Gamma \times V \to V$

such that for γ and v elements of Γ and V, respectively, $(\gamma, v) \mapsto \gamma \cdot v$, where the 'dot' represents the operation of γ on v, with the following properties:

(a). For each γ in Γ, the mapping $\rho_\gamma : V \rightarrow V$ defined by $\rho_\gamma(v) = \gamma \cdot v$ is linear; and

(b). If γ_1, γ_2 in Γ then $\gamma_1 \cdot (\gamma_2 \cdot v) = (\gamma_1 \gamma_2) \cdot v$.

Remarks:

(1). The 'dot' operation is not the binary operation that defines the group itself but is how the group is operating on the vector space, which may be independent of the group operation.

(2). Property (a) shows that the mapping defining the group action on the vector space V is a linear mapping. Furthermore, the mapping ρ that sends γ to ρ_γ, an element of GL(n), is called a *representation* of Γ on V, and therefore, necessarily, we will be discussing matrix representations acting on finite dimensional vector spaces. The mapping defining this matrix representation of Γ on V in definition (9) is a homomorphism.

(3). Property (b) states that the group action defined by the 'dot' is distributive.

Thus, the action and representation are essentially identical, but differ in point of view. The action shows how each element of the group Γ transforms each element of the vector space V, while the representation shows how the group transforms the whole space. For example, the group of rotations O(2) acting on $\boldsymbol{R}^2$ form the group of orthogonal linear operators that map $\boldsymbol{R}^2$ onto itself. These properties do not rely on a choice of basis, but upon choosing a suitable one, a matrix representation can be constructed. Since we are considering a finite dimensional vector space, we can construct such a matrix representation by defining how each element of O(2) transforms each basis element of the vector space. This is the same procedure as when computing a matrix representation of a linear transformation. Since groups have less structure than linear transformations, however, we view them as generalizations of linear transformations in this context.

We expect that since an infinite number of different basis sets may be chosen for any given vector space V, there may be differing descriptions of the same action, that is, in general, for any action, the representation is not necessarily unique. In this case, we say that two actions are isomorphic in the following sense:

<u>Definition 10:</u> Let V and W be vector spaces of equal dimension and assume that the group Γ acts on both V and W. These two spaces are Γ-*isomorphic* if there exists a (linear) isomorphism A that maps V to W such that $A(\gamma \cdot v) = \gamma \cdot (Av)$ for all v in V, γ in Γ.

<u>*Remark:*</u> Note that the action of γ on the left-hand side of the equation is on V while on the right it is on W. Therefore, the matrix A is essentially a change of coordinates that commutes with the action of the group Γ.

If we consider this isomorphism property in terms of a matrix representation, with matrix multiplication for the group operation and the operation of the isomorphism A, then this property can be expressed as,

$$(A\rho_\gamma)v = (\rho_\gamma A)v,$$

where A is now a matrix, and, as above, ρ_γ is the matrix representation of the group element γ, in GL(n). This clearly states that the two matrices A and ρ_γ commute. If A is an invertible matrix, then we have the usual similarity transformation acting on ρ_γ.

<u>Definition 11:</u> Let Γ be a Lie group acting linearly on a vector space V. A subspace W of V is called Γ-*invariant* if $\gamma \cdot w$ is in W for every w in W. A representation, or action, of Γ on V is *irreducible* if the only Γ-invariant subspaces of V are $\{0\}$ and V itself. A subspace W of V is said to be Γ-*irreducible* if W is Γ-invariant and the action of Γ on W is irreducible.

We now present an example presented in [10, p.34] to hopefully make some of these concepts more concrete. This example will be referred to in the next section as well for the same purpose.

Example:

Consider the group O(2) acting on the space $\mathbb{R}^3$. The group O(2) is comprised of rotations about a fixed axis described by the special orthogonal group SO(2) with determinant of +1, and equivalent rotations to these with determinant -1. Thus, there is an additional element in O(2) that is applied to the elements of SO(2) to change the sign of their determinant. This element will be called the *flip* element. Therefore, we need only to describe two elements in order to describe the group O(2). Let Θ be an element of SO(2), and let κ represent the flip. Define their actions on the real 3-space by:

$$\Theta \cdot (x, y, z) = (x\cos 2\theta - y\sin 2\theta, x\sin 2\theta + y\cos 2\theta, z),$$
$$\kappa \cdot (x, y, z) = (x, -y, -z),$$

with the obvious matrix representations:

$$\rho_\Theta = \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_\kappa = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We now consider the properties listed in definition 11, beginning with invariance. The rotation element rotates only the two-dimensional (x,y)-plane. Thus, if we define the subspace V_1 by:

$$V_1 = \mathbb{R}^2 \times \{0\} = \{(x, y, 0)\},$$

then the action of the rotations leaves this subspace invariant. Furthermore, note that the flip element also leaves this subspace invariant. Thus, this is one O(2)-invariant subspace. Additionally, note that the rotation element leaves the z-axis fixed, therefore define the subspace V_2 by:

$$V_2 = \{0\} \times \boldsymbol{R} = \{0,0,z\}.$$

Then the rotation element leaves this subspace invariant. Further, again, the flip leaves this subspace invariant. Thus V_2 is also an O(2)-invariant subspace.

Now consider the notion of irreducibility. Since any vector in the (x,y)-plane can be rotated to any position in the plane by an appropriate choice of the angle θ, and the flip action places any arbitrary vector in the (x,y)-plane into another quadrant of the plane, O(2) acts irreducibly on V_1 as well. The same holds true for the subspace V_2. We will refer back to this example in the following section.

Definition 12: A representation of a group on a vector space V is *absolutely irreducible* if the only linear mappings on V that commute with the group are scalar multiples of the identity.

Example:

Consider the group O(2) acting on the vector space $\boldsymbol{R}$. As in the above example, we need to define the actions of the rotation and the flip elements of O(2) on this space. We choose the following standard definitions given in terms of their matrix representations:

$$\rho_{SO(2)} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}, \qquad \rho_\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Now consider a general 2×2 matrix is of the form:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

By a direct computation, for the matrix A to commute with the matrix representation of SO(2), the following conditions must hold: a = d, c = -b. For the matrix A to commute with the flip matrix, by a direct computation, it is easy to see that the following conditions must hold: a = d, c = b. For both of these conditions to hold simultaneously, therefore, the only matrices that commute with the group O(2) acting on the space $\mathbb{R}^2$ are scalar multiples of the identity, and therefore O(2) is said to act absolutely irreducibly on this space. Absolute irreducibility has significance in terms of dynamical systems that will be shown in the next section The last definition of this subsection is perhaps the most important in the discussion of dynamics.

<u>Defintion 13:</u> The *orbit* of the action of Γ on an element x in V is the set $\Gamma x = \{\gamma \cdot x \mid \gamma \in \Gamma\}$. The *isotropy subgroup* of x in V is the set $\Sigma x = \{\gamma \in \Gamma \mid \gamma \cdot x = x\}$. The *fixed-point subspace* of a subgroup Σ of Γ is the set $\mathrm{Fix}(\Sigma) = \{x \in V \mid \sigma x = x \text{ for each } \sigma \in \Sigma\}$.

<u>Remarks:</u>

(1). The orbit of the action is simply the elements formed by the action of all the group elements on a particular vector in V;

(2). The isotropy subgroups are the group elements of Γ that do not alter (or act trivially on) a particular vector x in V, and hence it is said that the vector x *contains* the symmetry described by the isotropy subgroup;

(3). The fixed-point subspace is the collection of vectors that are fixed by any given subgroup of the group Γ acting on the vector space V.

<u>Example:</u>

Consider the example above of O(2) acting on $\boldsymbol{R}^3$ with the same actions defined above. Choose the vector x = (1,0,0). The orbit of this vector is computed by applying all elements of the group O(2) on the vector. In this case, this is fairly simple since the flip element acts trivially on this vector while the rotation element sends this vector to any arbitrary position in the (x,y)-plane. Thus, the orbit is the subspace V_1 defined above. Now choose the vector x = (0,0,1). The isotropy subgroup of this vector is SO(2) since it acts trivially on the z-axis. Finally, notice that the square of the flip is the identity. Thus, the flip element is a subgroup of O(2) that we shall denote Z_2. The fixed-point subspace of the subgroup Z_2 is the x-axis since the flip acts trivially on that direction. The fixed-point subspace of SO(2) is the subspace V_2 defined above, i.e., the z-axis since it acts trivially on that direction.

Again, the purpose of this subsection was to introduce the terminology that is used in the study of dynamics via the equivariant bifurcation theory. Though this discussion may be cryptic, the primary point to be made here is that group actions on a vector space behave similarly to linear transformations acting on a vector space. This is key to understanding understanding equivariant bifurcation theory, as we will present it.

2.2 Dynamical Systems and Symmetry

Our interest in the following discussion is in nonlinear, vector-valued, evolution equations of the form

$$\dot{x} = f(x,\lambda), \;\; x \in \boldsymbol{R}^n, \;\; \lambda \in \boldsymbol{R}, \;\; f: \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}^n, \tag{2.1}$$

where λ is a control or bifurcation parameter. In particular, we are interested in the equilibrium and bounded time-dependent solutions of the vector field f that "possesses" symmetry in some sense. The implications of how symmetry of the vector field affects the existence of equilibrium solutions is the primary result of equivariant bifurcation theory. There is some additional terminology required, and again we start with a few definitions.

<u>Definition 14</u>: A real-valued function $f : V \to \mathbb{R}$, V a vector space, is *invariant* under the action of a compact Lie group Γ if

$$f(\gamma \cdot x) = f(x),$$

for all γ in Γ, x in V. An *invariant polynomial* is a polynomial with this property. A function is said to be *equivariant* under, or to *commute* with, the action of Γ if

$$f(\gamma \cdot x) = \gamma \cdot f(x),$$

for all γ in Γ, and x in V. As a point of terminology, we will say that a vector field *possesses* a certain symmetry if it is equivariant under its action.

<u>Example:</u> The simplest example of the two properties in definition 14 is the action of the reflection group on a function of one real variable, f(x). This group is defined as the set $\{\pm 1\}$ with the binary operation of standard multiplication of two real numbers. If f(-x) = f(x) then the function is invariant under the action of this group and clearly the function is even. If f(-x) = -f(x), then the function is said to be equivariant under the action, and f is an odd function. Furthermore, if f is an invariant polynomial under this group action, then f must have the structure $f(x^2)$ while in the equivariant case, it must have the structure $xg(x^2)$ where g is therefore invariant. Thus, the terms invariant and equivariant have very different implications for the structure of the function f, and therefore the same holds true for a vector field. We have chosen a polynomial for this example not simply because it is an obvious choice for the discussion, but also since we will be restricting ourselves to polynomials throughout this proposal. This restriction does not pose a strong constraint on our problem since many system models are polynomial in nature, or can be reduced by a normal form procedure to a polynomial (we refer the reader to [10, 16] for a discussion of normal form reductions for problems with symmetry).

From the point of view of dynamical systems, if we consider an equivariant, vector-valued, polynomial evolution equation of the form given in equation (2.1), then by applying the action of a compact Lie group Γ on both sides of the equation we have:

$$\gamma \cdot \dot{x} = \gamma \cdot f(x) = f(\gamma \cdot x).$$

If we evaluate the vector field along an equilibrium solution, $\overline{x}$, then $\gamma \cdot (\dot{\overline{x}}) = 0 = f(\overline{x})$, and the equation above becomes:

$$0 = \gamma \cdot f(\overline{x}) = f(\gamma \cdot \overline{x}) = f(\overline{x}).$$

What we see here is that an equivariant vector field of this type automatically gives rise to an invariant vector-valued function when evaluated along an equilibrium solution. We may construct all such polynomials that are invariant under the action of a given compact Lie group by applying their actions on a general representation of a polynomial, and finding conditions on the coefficients to satisfy the invariance properties. We refer the reader to [10, pp. 43-5] for examples.

Another consequence of equivariance of a vector field as defined in equation (2.1) involves the equivariance identity,

$$f(\gamma \cdot x, \lambda) = \gamma \cdot f(x, \lambda). \tag{2.2a}$$

To discuss linear stability of equilibrium solutions, the Jacobian must be computed. Therefore, apply the chain rule to both sides of equation (2.2a) to derive the identity:

$$(df)_{\gamma \cdot x, \lambda} \cdot \gamma = \gamma \cdot (df)_{x, \lambda}. \tag{2.2b}$$

When equation (2.2b) is evaluated along the trivial solution, we see that,

$$(df)_{0,\lambda} \cdot \gamma = \gamma \cdot (df)_{0,\lambda}. \tag{2.2c}$$

Thus the Jacobian of an equivariant vector field, when evaluated along a trivial solution branch, for any value of parameter, must commute with the group action. Therefore, if the group Γ acts absolutely irreducibly on the vector space $\boldsymbol{R}^n$, in this case, then this Jacobian must be a scalar multiple of the identity matrix, or

$$(df)_{0,\lambda} = c(\lambda)I. \tag{2.3}$$

By definition 14 then, we must require that $c(0) = 0$ as well.

We now have enough terminology to present the two major results from equivariant bifurcation theory that are important for our purpose (both are presented in [10, p.82 and p. 83, respectively along with associated proofs]). These can be stated as follows:

<u>Theorem 1</u>: (The Equivariant Branching Lemma). Let Γ be a compact Lie group.

i. Assume Γ acts absolutely irreducibly on the finite-dimensional vector space $\boldsymbol{R}^n$;
ii. Let $f: \boldsymbol{R}^n \times \boldsymbol{R} \to \boldsymbol{R}^n$ be a Γ-equivariant vector field with $f(0,\lambda) = 0$. By (i), we also have $(df)_{0,\lambda} = c(\lambda)I$, where I is the $n \times n$ identity matrix, (df) is the Jacobian matrix with the subscripts denoting evaluation of the Jacobian along the vector $x = 0$ for arbitrary parameter λ. Assume $c(0) = 0$, the condition for a bifurcation to occur;
iii. Assume $c'(0) \neq 0$, the eigenvalue crossing condition; and
iv. Assume $\dim \mathrm{Fix}(\Sigma) = 1$ where $\Sigma \subset \Gamma$ is a subgroup.

Then there exists a unique branch of solutions to $f(\overline{x},\lambda) = 0$ emanating from $(0,\lambda)$ where the symmetry of the solutions is at least Σ.

If the conditions for this theorem are not satisfied, then we may satisfy ourselves with the following theorem:

Theorem 2: Let Γ be a Lie group acting on the finite dimensional vector space $\mathbb{R}^n$.

Assume:

a) $\mathrm{Fix}(\Gamma) = \{0\}$;

b) $\Sigma \subset \Gamma$ is an isotropy subgroup satisfying $\dim \mathrm{Fix}(\Sigma) = 1$; and

c) $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a Γ-equivariant bifurcation problem satisfying

$$(df)_{0,0}(v_0) \neq 0$$

where $v_0 \in \mathrm{Fix}(\Sigma)$ is nonzero.

Then there exists a smooth branch of solutions $(tv_0, \lambda(t))$ to the equation $f(x,\lambda) = 0$.

These two theorems tell us that the trivial solution branches to non-trivial fixed-point solutions that contain symmetry of a subgroup of the full group of the system corresponding to one-dimensional fixed-point subspaces. The advantage of theorem 1 is that it holds simultaneously for all subgroups of this type, whereas for theorem 2 to be applicable, the nondegeneracy condition $(df)_{0,0}(v_0) \neq 0$ must be shown for each of the subgroups. The advantage of theorem 2 is that it does not require absolute irreducibility of the group action on the vector space, thereby removing the constraint on the Jacobian of the vector field as a scalar multiple of the identity matrix. These two theorems say nothing about Hopf bifurcations and only deal with fixed-point solutions. Therefore, separate stability analysis must be performed to determine if any time-dependent solutions branch from the trivial solution.

These two theorems suggest that, if we know all of the isotropy subgroups of an absolutely irreducible action, then we know all of the possible equilibrium solutions of the system and that symmetry-breaking bifurcations connect them together in some fashion. If the action is not absolutely irreducible, we still know a large subset of the total number of possible equilibrium solutions by finding these subgroups. Knowing the solutions alone, however, does not give a complete description of dynamics since stability analysis determines how these solution classes connect to one another. Thus, stability analysis is an important component of the overall analysis. To illustrate the possible conncections, we order the isotropy subgroups into a lattice structure where each level of this lattice is defined by the dimension of the corresponding fixed-point subspace for each subgroup, and the branches are defined by the subgroup structure. This construct is known as the isotropy subgroup lattice, and we shall see in chapter 4 how to produce such a lattice in detail for the system at hand. We believe that the tools of equivariant bifurcation theory can play an important role in our understanding of the nonlinear dynamics of the weakly coupled identical oscillators. Thus, we proceed in the rest of this work, based on the following assumption:

Working Hypothesis: If the conditions for theorems 1 or 2 above are satisfied then we have some knowledge of the dynamics in relation to the isotropy subgroup lattice, but in the event that these conditions are not met, the lattice is still an important classification scheme for the dynamics of an equivariant vector field.

On this assumption, we begin our analysis of weakly coupled, forced mechanical oscillators. We first present the motivating example in the next chapter.

CHAPTER 3
WEAKLY COUPLED MECHANICAL OSCILLATORS

We choose the simplest physical example of nonlinear, coupled mechanical oscillators to introduce the dynamics of such systems. This example considers each oscillator to be excited by a sinusoidal force that lies in the plane of the ring containing the oscillators. The nonlinear oscillators are coupled via linear springs to their nearest neighbors. Though this example is simple, it contains all the essential features we require to place the analysis in a physical context. The first section introduces this model and derives the equations of motion for weakly nonlinear dynamics. An application of the averaging procedure reduces the equations to an autonomous dynamical system. Since the system model includes an external force driving the system, the resulting averaged equations contain the influence of forcing through nonhomogeneous terms. A modification of this model is also introduced that results in parametric forcing of each oscillator. The second section generalizes the equations derived in the example by simply exploiting the symmetry of such a system, and it is here that the group structure is discussed in detail. Both parametric and direct forcing are discussed in this context. Through this approach, we show that the averaged equations of motion derived in section 1 are simply a low order (polynomial) normal form for the general case of cyclically coupled, and harmonically excited mechanical systems.

3.1 Motivational Example

The system under consideration is taken from [5] where a cyclic system of n identical particles of mass m attached to ground by nonlinear torsion springs, and arranged in a ring by nearest-neighbor coupling via weak linear extensional springs, is considered. As per figure 1, let x_i be the transverse displacement of the i^{th} particle, considered small, and assume the linear coupling spring force to be of the form:

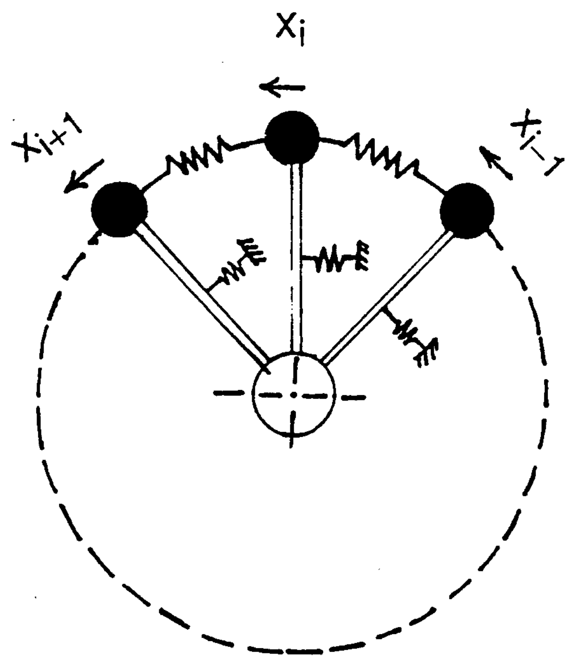


Figure 1. The nonlinear cyclic system

$$f_{ij} = \varepsilon k(x_i - x_j). \tag{3.1}$$

Here, j is (i + 1) or (i - 1) due to the assumption of nearest-neighbor coupling, and ε, $0 < \varepsilon << 1$, is a small order parameter. The ground spring is assumed to have weak nonlinearities, and therefore, it exerts a force (providing the equivalent torque) of the form:

$$\Gamma_i = T_1 \frac{x_i}{a^2} + \varepsilon T_2 \frac{x_i^3}{a^4}, \tag{3.2}$$

where a is the radius of the ring geometry, and T_1 and T_2 are the linear and cubic torsional stiffnesses. We also assume weak viscous damping of the form $\varepsilon \bar{d} \dot{x}_i$. Thus, the equations of motion for the i-th mass particle may be written in the form

$$m\ddot{x}_i + \frac{T_1}{a^2} x_i + \varepsilon f_i\left(x_{i-1}, x_i, x_{i+1}\right) = 0, \tag{3.3}$$

with

$$f_i\left(x_{i-1}, x_i, x_{i+1}\right) = k\left[\left(x_i - x_{i-1}\right) + \left(x_i - x_{i+1}\right)\right] + T_2 \frac{x_i^3}{a^4} + \bar{d}\dot{x}_i. \tag{3.4}$$

By introducing a new time scale $\tau = \overline{\omega} t$, where $\overline{\omega} = \left\{T_1 / ma^2\right\}^{1/2}$, the equations governing the response of the complete system may be written in the form,

$$x_i'' + x_i + \left(\varepsilon / m\overline{\omega}^2\right) f_i = 0, \quad i = 1, 2, \ldots, n, \tag{3.5}$$

where prime denotes differentiation with respect to the nondimensional time τ. We include a weak external forcing function in the system by writing equations (3.5) in a matrix notation, and applying $F^{ex} = \overline{\mu}\Gamma \cos \overline{\Omega} t$ to the system, where $\overline{\mu}$ represents the amplitude of the forcing while Γ denotes the spatial distribution, whereby the equations of motion become:

$$\underline{x}'' + \underline{x} + (\varepsilon/m\overline{\omega}^2)(\underline{F} - \underline{F}^{ex}) = 0. \qquad (3.6)$$

The external forcing is restricted to be in the plane of the ring system.

In order to find periodic solutions for the system of equations (3.6), we apply the method of averaging with respect to the external forcing frequency, $\Omega = \overline{\Omega}/\overline{\omega}$, by assuming the response of the system to be of the form:

$$\underline{x} = \sum_{j=1}^{n} A_j \underline{U}_j \cos(\Omega\tau + \alpha_j), \qquad (3.7)$$

where $\underline{U}_j$ is simply a column vector of zeros except for a 1 in the j^{th} place, and A_j and α_j are the amplitude and phase of the j^{th} oscillator, respectively. Substituting equations (3.7) into equations (3.6), introducing the notation $\overline{\mu}_j \equiv \underline{U}_j \cdot \overline{\mu}\Gamma$, which represents the external forcing on the j^{th} oscillator, perturbing the response frequency away from resonance by setting $\Omega^2 = 1 - \varepsilon\overline{\lambda}$, and averaging the resulting equations over the period $T=2\pi/\Omega$, we obtain the following averaged equations (see [5] for details):

$$\begin{aligned} A_j' &= -\varepsilon\left\{\kappa\left[A_{j-1}\sin(\alpha_j - \alpha_{j-1}) + A_{j+1}\sin(\alpha_j - \alpha_{j+1})\right] + dA_j + \mu_j \sin\alpha_j\right\}, \\ A_j\alpha_j' &= -\varepsilon\left\{-\kappa\left[A_{j-1}\cos(\alpha_j - \alpha_{j-1}) + A_{j+1}\cos(\alpha_j - \alpha_{j+1}) - 2A_j\right] + pA_j^3 \right. \\ &\qquad \left. -\lambda A_j - \mu_j \cos\alpha_j\right\}, \end{aligned} \qquad (3.8)$$

$j = 1,..,n$, where the parameters κ, d, μ_j, p, and λ are dimensionless combinations of the parameters introduced above, and are given by:

$$\kappa = \bar{k}/2m\bar{\omega}^2\Omega,\ d = \bar{d}/2m\bar{\omega}^2,\ \mu_j = \bar{\mu}_j/2m\bar{\omega}^2\Omega,\ p = T_2/4m\bar{\omega}^2\Omega,\ \lambda = \bar{\lambda}/2\Omega.$$

Thus, κ is a dimensionless stiffness for the coupling springs and is always positive; d is a dimensionless damping parameter that is positive since the inherent negative sign has already been taken into account in the equations of motion; μ_j is the dimensionless external forcing amplitude on oscillator j; p is the dimensionless cubic nonlinearity in the ground springs with the property that it is positive when the spring is hardening, whereas it is negative for a softening spring; and λ is the dimensionless detuning frequency with no sign restriction. Also note that, due to its definition, the detuning λ is positive when the excitation frequency is detuned below the resonance or natural frequency, while it is negative when the excitation frequency is above the natural value.

At this point, we depart from the discussion in [5] and transform equations (3.8) into a form more convenient for our subsequent discussion by defining the complex variables $z_j = A_j \exp(i\alpha_j)$, j = 1, 2, ..., n. This results in the system of equations:

$$z_j' = -dz_j - i\left\{\left[\lambda - p|z_j|^2\right]z_j + \mu_j + \kappa(z_{j+1} + z_{j-1} - 2z_j)\right\}, \tag{3.9}$$

j = 1,..,n, where a prime now denotes differentiation with respect to the rescaled time $\varepsilon\tau$. This system of n equations forms a normal form for the dynamics of n weakly coupled identical oscillators in 1:1 resonance which are subject to a weak, external, planar, and resonant harmonic forcing. Notice that the vector field is polynomial, and that the only off-diagonal components arise due to the coupling. Therefore, in subsequent development, we refer to the conservative portion of these diagonal terms as the "uncoupled internal mode model", while the coupling terms will be referred to as the "coupled vibration model". The forcing and damping terms will be referred to as the "external excitation model" and the "damping model", respectively. When averaging is applied to the original equations of motion (3.6), the external forcing becomes a parameter. Note however that it has its own set of initial conditions independent from the amplitudes and phases of the oscillators. We quantify this below. The external forcing parameters μ_j are shown to be real quantities due to the zero fixed phase in the assumed form of the external forcing when the averaging takes place. If this fixed phase value is assumed to be nonzero, the parameters μ_j become complex quantities.

In the uncoupled case ($\kappa = 0$), the system consists of n resonantly forced Duffing oscillators (Nayfeh and Mook [17]) in 1:1 resonance, and therefore, in the case of hardening springs ($p > 0$), we are guaranteed three solutions for each oscillator in a frequency interval above ($\lambda < 0$) the resonant frequency. It is well-known (see Nayfeh and Mook [17]) that for this case, the solution with the largest amplitude is a continuation of the solution existing below the resonant frequency, while, for damping values satisfying $d < \left(p\mu^2/2\right)^{1/3}$, the other two branches of solutions arise via a turning point and have opposing stability characteristics. Therefore, these three coexisting solutions have differing stabilities. For the coupled system, equations (3.9), it can be shown [5] that solutions are asymptotically bounded by a sphere whose size is determined by the magnitude of the external forcing and the damping present in the system. Also note that, since each spring-mass system is in external resonance, it must maintain a fixed phase with respect to external forcing. If the phase of oscillation of each mass is changed, the forcing phase must also be changed by the same amount.

Parametric Excitation Case. Another example of some interest in this study is the case of a system with a fixed radial force acting on each oscillator. Since we wish to perform this analysis with an eye towards turbomachinery applications, the external forcing acting on each oscillator can be viewed to have multiple components: The circumferential component used in the above derivation, a radial component, and a component that acts perpendicular to the plane of the ring. In the present model we have restricted our discussion to forces confined to the plane of the system. To describe the radial component of forces acting on the system, we consider the same physical system where the mass is only allowed to oscillate circumferentially, thereby placing a constraint on the system. As in the circumferential forcing case, we are considering small amplitude responses, and thus the component of the radial force in the equations of motion can be written as $F_j(t)x_j$ for each $j = 1, 2, ..., n$. Now, assuming the radial forcing function to be of the form $F_j(t) = \mu_j \cos(2\Omega t)$, assuming the response in the form of equations (3.7), and then performing the same averaging procedure, a system of averaged equations similar to equations (3.8) is obtained. In complex form, as per equations (3.9), this system contains the term $i\mu_j z_j$ rather than the direct forcing term $i\mu_j$. Thus, this forcing, rather than being a direct forcing, is a parametric forcing, placing the system in parametric resonance with the ratio of response to forcing frequencies of 1:2. If the phase of this parametric forcing is assumed to be nonzero, then, as in the direct forcing case, the coefficient μ becomes a complex parameter in the normal form equations rather than being a purely real quantity.

In axial flow compressors, the external excitation may have a blade-to-blade phase difference due to the intake fluid flow encountering bolts or rivets, for example, placed cyclically around the chamber. The forcing in this case would be of equal magnitude and frequency, as per our model, but the phase would be distributed in a cyclic manner. This effect in the turbomachinery literature is known as "engine order excitation", and we will see in chapter 4 below that this is a generic effect in cyclic systems of the type considered here.

We now discuss the symmetry structure of our system in detail using this motivational example, and generalize the system in equations (3.9) leaving the system model as general as possible.

3.2 Symmetry, Invariant Polynomials and Modeling

In this subsection, we wish to generalize the model in the above motivational example by considering polynomial functions that possess the symmetry of the system under investigation above. We will consider each of the components of the system model in turn, that is the "uncoupled internal mode model", the "damping model", the "coupling model", and the "forcing model". A detailed discussion of the symmetries and corresponding groups will be given to derive these models, and this will naturally lead us into the discussion of the isotropy subgroup lattice in chapter 4. We begin with the uncoupled internal mode and damping models.

3.2.1 Uncoupled Internal Mode and Damping Models

The group of interest at this point is the special orthogonal group SO(2) with the usual defintion:

$$SO(2) = \langle P \in GL(n):\ P^t P = I, \det P = 1 \rangle. \tag{3.10}$$

Since the free vibration model form we wish to pursue contains only a single degree-of-freedom, we need only discuss the 2×2 matrices in GL(2) that satisfy this definition. For the planar system, we may choose a complex variable representation by rewriting the two real variables (amplitude and phase) into a single complex variable resulting in the following representation:

$$SO(2) = \langle e^{i\theta}:\ \theta \in \boldsymbol{R} \bmod 2\pi \rangle. \tag{3.11}$$

In this representation, the group corresponds to phase translations of a complex quantity, z, by the amount θ. In the mathematics literature, the group SO(2) written in this form is sometimes denoted T^1 and we will refer to it in this way. It can be shown by a direct calculation, as per Golubitsky et al. [10], that all T^1-invariant polynomials are given by:

$$h(z,\bar{z}) = -i\left[p\left(|z|^2\right)z + iq\left(|z|^2\right)z\right]. \tag{3.12}$$

Comparing this equation with equations (3.9) shows that the damping model is given by the polynomial q, while the uncoupled internal mode model is contained in the polynomial p. As seen above, the presence of damping guarantees bounded motions for the system and, therefore, we require the function q to be negative definite to satisfy this criterion. Without the damping polynomial, the symmetry of this component of the system would be O(2) (see [10]), and would complicate the discussion below in chapter 4. Note also that in physically realistic situations, all sub-structures contain some damping no matter how small it may be. Thus, we exclude the O(2) case from our consideration. We wish to couple this T^1 system to itself n times with a coupling model that also satisfies the conditions of being a T^1-invariant polynomial.

3.2.2 Coupling Model

Again, we wish to choose the simplest form for this coupling, and, thus we assume nearest-neighbor coupling as given in equation (3.9) and figure 1. The resulting system now has the symmetry of an n-gon, the strength of each coupling being equal. The coupling itself can be elastic or dissipative as long as it is nearest-neighbor and the coupling strength is equal. In the present study, we assume elastic coupling as shown in equations (3.9). If the coupling also contains a dissipative element, the coupling constant κ would be replaced with $i\kappa$.

The symmetry group that describes the symmetry of an n-gon is called the dihedral group, denoted by D_n, and has the definition:

$$D_n = \left\langle (\sigma,\kappa) : \sigma^n = \kappa^2 = 1, \sigma\kappa\sigma = \kappa \right\rangle. \tag{3.13}$$

This group consists of two elements with slightly different properties that when combined together as maps, generate all the elements of the set. We will refer to such elements as *generators* since they generate all elements of the group. Another example of generators are the rotation and flip elements of the group O(2) discussed earlier. These generators are σ, a cyclic group element, and κ, a flip element, and we discuss each in turn. The discussion given here is based largely on [14] where the authors have used the same group actions and representations for their discussion, but we include a few minor corrections.

A cyclic group is a single element group that consists of a map that has the property, as stated in equation (3.13), that n multiplications of this element results in the identity. The action of this generator on a set of n elements is as a shift operator, that is, given a set of elements $\{z_1, z_2, ..., z_n\}$ the action is given by:

$$\sigma\{z_1, z_2, \ldots, z_{n-1}, z_n\} = \{z_n, z_1, z_2, \ldots, z_{n-1}\}, \tag{3.14}$$

which is just a cyclic shift of n elements. The representation of this element is a matrix with the property that it shifts the elements of n-vectors in this same cyclic manner. Therefore, since we are considering left multiplication for our matrix representation, the matrix representation for the cyclic shift element is an $n \times n$ matrix with the subdiagonal populated with one's, the (1,n) element of the matrix being unity, and the rest of the entries zero. Note that, if we had right multiplication as our binary operation, this representation would be to shift the columns in this manner. It is an easy matter to show that this generator forms a group unto itself, and to help understand the isotropy subgroup table that we will show in the next section, we state the following theorem, without proof, that shows some essential properties of these single element, cyclic groups (see [18, pp. 105-6] for the proof).

<u>Theorem 3:</u> Let $G = \{ \sigma \mid \sigma^k = 1 \}$, the cyclic group of order k.

i. Let H be a subgroup of G. Then H is a cyclic group with generator $x = \sigma^m$, where m is the least positive power of σ which lies in H, or else $H = \{1\}$. If k is finite then m divides k and the order of H is k/m. If the order of G is infinite then H is infinite or H = {1}.

ii. Conversely, if m is any positive integer dividing k, then the set S generated by the element x as in (i) is of order k/m. Consequently there is a cyclic subgroup of order q for any q that divides k.

iii. The number of distinct subgroups of G is the same as the number of distinct divisors of k for finite k.

iv. There is at most one subgroup of G of any given order for G finite.

This theorem essentially states that if we can find all binary factorizations of the cycle order n, then we know all the subgroups that exist.

The other generator for the dihedral group is the flip operation, κ. The action of the flip on a vector in $\mathbb{C}^n$ is chosen to be:

$$\kappa\{z_1,z_2,z_3,\ldots,z_{n-1},z_n\} = \{(z_1)(z_2,z_n)(z_3,z_{n-1})\ldots(z_{[(n+1)/2]},z_{n-[(n+1)/2]})\},$$

where the square bracket notation [m/2] denotes the integer part of m/2, and the curved braces denote the action of interchanging the two elements enclosed. For this action, if n is even, then the first and (n+2)/2 element are fixed by the flip while if n is odd then only the first element is fixed. For example, for five elements, the action of κ is given by,

$$\kappa\{z_1, z_2, z_3, z_4, z_5\} = \{z_1, z_5, z_4, z_3, z_2\},$$

while for six elements the action is

$$\kappa\{z_1, z_2, z_3, z_4, z_5, z_6\} = \{z_1, z_6, z_5, z_4, z_3, z_2\}.$$

Thus, the representation should be constructed so as to fix the first row of a matrix and produce this sort of rearrangement among the other rows. Thus, the representation for the flip κ has the structure:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & ⋰ & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}. \tag{3.15}$$

Note that the square of this matrix is the n dimensional identity matrix. The choice of which element is the first element of the set $\{z_1,z_2,\dots,z_n\}$, that remains fixed, is arbitrary and may be chosen for convenience, but the relative order of the elements must be respected. This completes the discussion of the individual symmetries and their associated groups that are present in the system. Now we wish to combine them into one action, and give the corresponding representation.

Again, taking from [14], the action we choose is:

$$(\rho, e^{i\theta})(z_j) = e^{i\theta} z_{\rho(j)}, \; j = 1,\dots,n, \; \theta \in [0,2\pi).$$

Thus, each element z_j, $j = 1,\dots,n$, is reordered by some element of the dihedral group, ρ, and, simultaneously, its phase is translated by the amount θ. Hence, the matrix representation of the action of the T^1 group is that of a diagonal matrix with the nonzero elements given by $e^{i\theta}$, thereby, translating the phase of all n elements by the same amount. Thus, the symmetry group of the system may be written as $D_n \times T^1$ as the two group operations are independent but act simultaneously. The representation then, is:

$$\gamma(\underline{z}) \mapsto e^{i\theta} I_n A \underline{z}, \;\; \gamma \in D_n \times T^1, \;\; \underline{z} \in \mathbb{C}^n, \tag{3.16}$$

where A is the representation of D_n. In light of this group construct then, the system of equations we wish to study is of the form:

$$z_j' = -i\left\{p\left(z_j\bar{z}_j\right)z_j + iq\left(z_j\bar{z}_j\right)z_j + h\left(z_{j+1} - z_j, z_{j-1} - z_j\right)\right\}, \tag{3.17}$$

j = 1,..,n. The polynomials p, q, and h will be assumed to have real coefficients. Physically, having complex coefficients in the polynomials p and q would correspond to a delay in the dynamics of the system due to the application of some excitation process, while, as discussed above, complex coefficients in the polynomial h correspond to dissipative elements in the coupling. These are unnecessary complications. Again, we assume that the coupling strength of the oscillators is equal in both directions. At this point, this set of equations does not explicitly contain the effect of the external excitation, to which we now turn.

3.2.3 External Excitation Model

First, consider parametric forcing. In this case, the forcing enters as a multiplicative factor to each of the scalars z_j, j = 1, 2, ..., n. Furthermore, recall that in the motivational example, it came about as the first term in the Taylor expansion of the sine trigonometric function. Therefore, if more terms are kept in this expansion, an odd order polynomial would result in the averaged equations, resulting in the same form as for the uncoupled internal mode model polynomial p. Thus, in this case, we assume that we have another polynomial of this type describing the parametric forcing with the appropriate multiplicative parameters being real for each term. In the remainder of this section, we will concentrate on the direct forcing case.

To maintain the symmetry of the system, we assume that the external excitation model possesses the same symmetry (T^1) as the uncoupled internal mode model, and it acts identically on each oscillator so as to respect the $D_n \times T^1$ group structure of the system. Therefore, the polynomial representing the forcing must have the same form as given in equation (3.12). The damping component of this polynomial does not correspond to a physical excitation process of interest and is dropped from the model. We need to augment the space $\mathbb{C}^n$ of the vector field for the oscillators to include the variables defining the external excitation. Thus, we have the following structure: $(z_1, z_2, ..., z_n; \mu_1, \mu_2, ..., \mu_n)$, where the z's contain information with respect to the oscillators' response to the external forcings μ's. We may finally rewrite the model representing the dynamics of the system (3.17) as:

$$z_j' = -i\left\{p\left(z_j\bar{z}_j\right)z_j + f\left(\mu_j\bar{\mu}_j\right)\mu_j + iq\left(z_j\bar{z}_j\right)z_j + h\left(z_{j+1} - z_j, z_{j-1} - z_j\right)\right\}, \tag{3.18}$$

$j = 1, 2, ..., n$, and augment it with the vector field description of the forcing, given by:

$$\mu_j' = w\left(\mu_j\right), \tag{3.19}$$

$j = 1,..,n$. Thus, we have a vector space of dimension 2n with a vector field structure that is a direct sum of equations (3.18) and (3.19). To simplify this model as much as possible, and to still maintain the spirit of the motivating example given above, we assume that $w = 0$ for all j, thereby making the direct forcing a parameter in the system (3.18). We wish to study the system near resonance since the largest as well as the most interesting response of structures arises for this condition. We can also visualize the situation as if the forcing is periodic with a Fourier spectrum such that one of the Fourier frequencies is in resonance with the natural frequency. The group representation for this augmented space is then given by the following structure:

$$\gamma \cdot \left(\underline{z}, \underline{\mu}\right)^T \mapsto e^{i\theta}\left(I_{2n}\right)(A \oplus A)\left(\underline{z}, \underline{\mu}\right)^T, \ \gamma \in D_{2n} \times T^1, \ \underline{z} \in \mathbb{C}^n, \ \underline{\mu} \in \mathbb{C}^n, \tag{3.20}$$

where A is the representation of the portion of γ corresponding to D_{2n}, the notation $\oplus$ denotes direct sum, and superscript T represents the transpose operation. Thus, the phase translation corresponding to the T^1 symmetry group acts equally on the z's and μ's, as does the representation of the dihedral group action. An external excitation component giving rise to direct forcing, assures, as per the above discussion in the motivational example, that the trivial solution (i.e. $(z_1, z_2, ..., z_n) = (0, 0, ...,0)$) for the oscillator's response does not exist, while, coupled with damping, an attracting periodic motion is guaranteed.

With this formulation of the system equations, we move on to study the dynamics of the system by first discussing the isotropy subgroup lattice. For the augmented structure of the vector field, it is appropriate to discuss equilibrium solutions corresponding to the isotropy subgroups by stating them for the augmented vector $(z_1, z_2, ..., z_n; \mu_1, \mu_2, ...,\mu_n)$. However, the external excitation terms are taken as parameters due to the averaging of an assumed harmonic excitation and therefore, their equilibrium solutions come directly from the spans of the fixed-point subspace spans. Thus, stating these solutions is a trivial matter and we restrict our attention to the oscillator responses. Physically, in the case of steady-state response of forced oscillators, the forcing phase determines the oscillator response phase, though in damped systems there may be a fixed phase lag between the oscillator response and the forcing. As discussed above, the phase relationships among the oscillator responses is the same as that for the forcing phases, and the augmented vector field construct presented here maintains this (physical) relationship.

CHAPTER 4
THE ISOTROPY SUBGROUP LATTICE AND DYNAMICS

The isotropy subgroup lattice will be used to obtain information on fixed-point solutions of the averaged system of equations, that then correspond to periodic motions in the original system. Thus, we first discuss in detail the structure of the isotropy subgroup lattice, its significance, and how to construct it for a given number of oscillators. The second section is devoted to explicit construction of the solution classes corresponding to the subgroups contained in the lattice, and the third section discusses briefly the difference between linear and nonlinear coupled oscillators.

As noted above, Ashwin and Swift [14] have used the same group action and representation for a $D_n \times T^1$-equivariant system with weak coupling. They derive the isotropy subgroup lattice that we use here, to classify the motions corresponding to each subgroup in terms of our system of equations. However, their system does not include damping or forcing, and therefore they only consider frequency equations. Furthermore, the amplitudes are ignored, and thus the existence of multiple solutions for each frequency value in the uncoupled case for each oscillator is not considered. We use the motivational example of the previous chapter to interpret the dynamics in terms of this mechanical system.

From [14], we have the following theorem, stated without proof:

Theorem 4: The isotropy subgroups of D_n x T^1 and their fixed-point subspaces are defined in table 1, where mk = n runs through all binary factorizations of n. The list constructed in this way has no duplications.

Immediately, we see in this theorem the importance of the binary factorizations of the cyclic group order shown in theorem 3. The problem with using this theorem, for our analysis is that the proof used in [14] relies on some of the structure they have imposed on their system that does not exist in ours. We believe that their proof can be modified appropriately, and the theorem suitably restated, so that it is applicable to our system, at

Table 1: The isotropy subgroups of $D_n \times T^1$. There are m blocks of k adjacent oscillators in the fixed-point spaces, where n = mk. As always, $\omega = \exp(i2\pi/n)$.

Isotropy Subgroup Σ	dim(fix(Σ))	fix(Σ)
k = 1		
D_n	1	(a,...,a)
$D_n(+-)$, for n even	1	(a, -a, a,...,-a)
$Z_n(p)$, $p \in \{1,...,[(n-1)/2]\}$	1	$(a, \omega^p a, \omega^{2p} a,..., \omega^{-p} a)$
k = 2		
$D_{n/2}(+-)$, for n = 0 mod 4	1	(a, a, -a, -a,...,a, a, -a, -a)
$D_{n/2}(\kappa)$	2	(a, b, a, b,...,a, b)
$Z_{n/2}(p)$, $p \in \{1,...,[n/4]\}$	2	$(a, b, \omega^{2p} a, \omega^{2p} b,..., \omega^{-2p} a, \omega^{-2p} b)$
k odd, $k \neq 1$		
D_m	(k+1)/2	(a, b, c, c, b,..., a, b, c, c, b)
$D_m(+-)$, for m even	(k+1)/2	(a, b, c, c, b,..., -a, -b, -c, -c, -b)
Z_m	k	(a, b, c, d, e,..., a, b, c, d, e)
$Z_m(p)$, $p \in \{1,...,[m/2]\}$	k	$(a, b, c, d, e, \omega^{5p} a, \omega^{5p} b,..., \omega^{-5p} d, \omega^{-5p} e)$
k even, $k \neq 2$		
$D_m(\kappa)$	k/2 + 1	(a, b, c, d, c, b,...,a, b, c, d, c, b)
$D_m(\kappa\sigma)$	k/2	(a, b, c, c, b, a,..., a, b, c, c, b, a)
$D_m(--)$	k/2	(a, b, c, -c, -b, -a,..., a, b, c, -c, -b, -a)
$D_m(+-)$, for m even	k/2	(a, b, c, c, b, a,..., -a, -b, -c, -c, -b, -a)
Z_m	k	(a, b, c, d,...,a, b, c, d)
$Z_m(p)$, $p \in \{1,...,[m/2]\}$	k	$(a, b, c, d, \omega^{4p} a, \omega^{4p} b,..., \omega^{-4p} c, \omega^{-4p} d)$

least in the direct forcing case. On this assumption, this chapter begins by describing the notation of the table, and how the table is used to form the lattice.

4.1 The Lattice

If we call the shift element of the dihedral group σ, and the flip element κ, then, as hown in [14], we have the following definitions of the various subgroups shown in the table:

$$D_m(\kappa) \equiv \langle\{\sigma^k, \kappa\}\rangle, \tag{4.1}$$

$$D_m(\kappa\sigma) \equiv \langle\{\sigma^k, \kappa\sigma\}\rangle, \tag{4.2}$$

$$Z_m \equiv \langle\{\sigma^k\}\rangle, \tag{4.3}$$

$$D_m(+-) \equiv \langle\{(\sigma^{k-1}\kappa,1),(\kappa\sigma,-1)\}\rangle, \text{ for m even,} \tag{4.4}$$

$$D_m(--) \equiv \langle\{(\sigma^{k-1}\kappa,-1),(\kappa\sigma,-1)\}\rangle, \text{ for k even,} \tag{4.5}$$

$$Z_m(p) \equiv \langle\{(\sigma^k,\omega^{pk})\}\rangle \text{ where p is an element of the set } \{1,\ldots,[m/2]\}, \tag{4.6}$$

where $mk = n$, and as above, the bracket notation $[m/2]$ denotes the integer portion of the division, and the element listed as ω represents the quantity $e^{i\theta}$, the phase translation element. In this notation, each element or elements grouped by the curved braces are generators for the subgroup. If there is more than one element listed as a generator, such as $(\sigma^{k-1}\kappa,-1)$, then this is a generator of two elements where the first element $\sigma^{k-1}\kappa$ is an element of the group D_n, while $e^{i\pi} = -1$ is an element of the group T^1. Groups (4.4-6) are formed by these two element generators, and in the mathematics literature, these are referred to as *twisted subgroups*. Groups (4.1-3) are therefore known as *untwisted subgroups*. Note that when k is odd for the subgroup in equation (4.2) it can be written as D_m since the additional flip action is trivial in that case. We now describe each of these classes of subgroups.

For the untwisted subgroups, equations (4.1-3), we see the property of binary factorization of an integer giving rise to subgroups, as shown in theorem 3, becoming apparent. The group given in equation (4.1) is the same as the definition of the dihedral group except that instead of single elements being identical as in the D_n case, blocks of k elements are identical. The same is true for the subgroup in equation (4.2) except that, as seen in the column listing the fixed-point subspace span in the table, the internal structure of each block is different. The third subgroup, in equation (4.3), has no flip action as part of its generator list and therefore is a pure cyclic shift of order m. Thus, we again have k identical blocks of m oscillators with a different internal structure as shown in the table. These untwisted subgroups form a class of motions such that all blocks oscillate in-phase with one another.

For the twisted subgroups, we again have m identical blocks of k oscillators, except that now we see the evidence of engine order excitation. In the case of equations (4.4) and (4.5) we have identical blocks but each block oscillates 180 degrees out of phase with its neighboring blocks, and each equation gives slightly different internal structure of each block. The third subgroup in equation (4.6) contains all the engine order excitations possible in the oscillator assembly. Thus, there are m blocks of k identical oscillators such that each block is excited out of phase by an integer multiple of the fundamental angular displacement around the ring, depending on the ordering present in the external forcing.

Thus we have three basic classes of dynamics based on these subgroups:

1. m blocks of k identical oscillators (or under particular combinations of the binary factorization of n, (k+1)/2, k/2 + 1, or k/2 oscillators) oscillating in phase, that are described by the untwisted subgroups. We will refer to this type of motion as "in-phase oscillations", denoted IP;
2. m blocks of k identical oscillators (and variations on the value k as per the IP case above, due to particular combinations of the binary factorization) oscillating 180 degrees out of phase, that are described by the twisted subgroups shown in equations (4.4) and (4.5). We will refer to this type of motion as "standing waves" and denote it SW; and
3. Engine order excitation of m blocks of k identical oscillators described by the twisted subgroup defined in equation (4.6). We will refer to this type of motion as "traveling waves" and denote it by TW. We will denote the excitation responsible for these motions as EO excitation.

To describe the isotropy subgroup lattice structure, we must define how the levels and branches are determined. The levels of the lattice are determined by the dimension of the corresponding fixed-point subspace, and from the table we see that this value depends on the number k in the binary factorization n = mk. Since the highest level is for the one dimensional subspace, we start with m = n and k = 1 for all cases, and end with m = 1 and k = n. The simplest example is that of n = 2. In this case, the binary factorizations, with the lattice structure in mind, is:

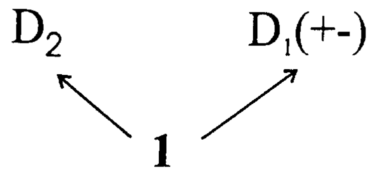
$$2 = (2)(1) = (1)(2),$$

and the lattice is shown in figure 2a. Thus, we have m = 2 and k = 1 which corresponds to the first subgroup listed in the table, D_2, and for m = 1 and k = 2, we have the fourth entry $D_1(+-)$, and the sixth $Z_1(1)$. Both of the subgroups D_2, D_1 have the same dimension of unity for the fixed-point subspace. Therefore, both of these subgroups form the top level of the lattice and no connection exists between them as one is not a subgroup of the other. In all cases, the last subgroup to be listed in the lattice is the solution where all oscillators have differing solution values which we will refer to as the trivial subgroup and denote by **1**. This subgroup is isomorphic to the subgroups Z_1 and $Z_1(1)$ listed in the table. Thus for the n = 2 case we see that we have the IP solution and the SW motion connecting to the trivial subgroup where both oscillators have differing solutions.

Since 2 is the only even prime number, the next simplest example is that of a prime number not equal to 2. In this case, let n be prime, then we have the following binary factorization:

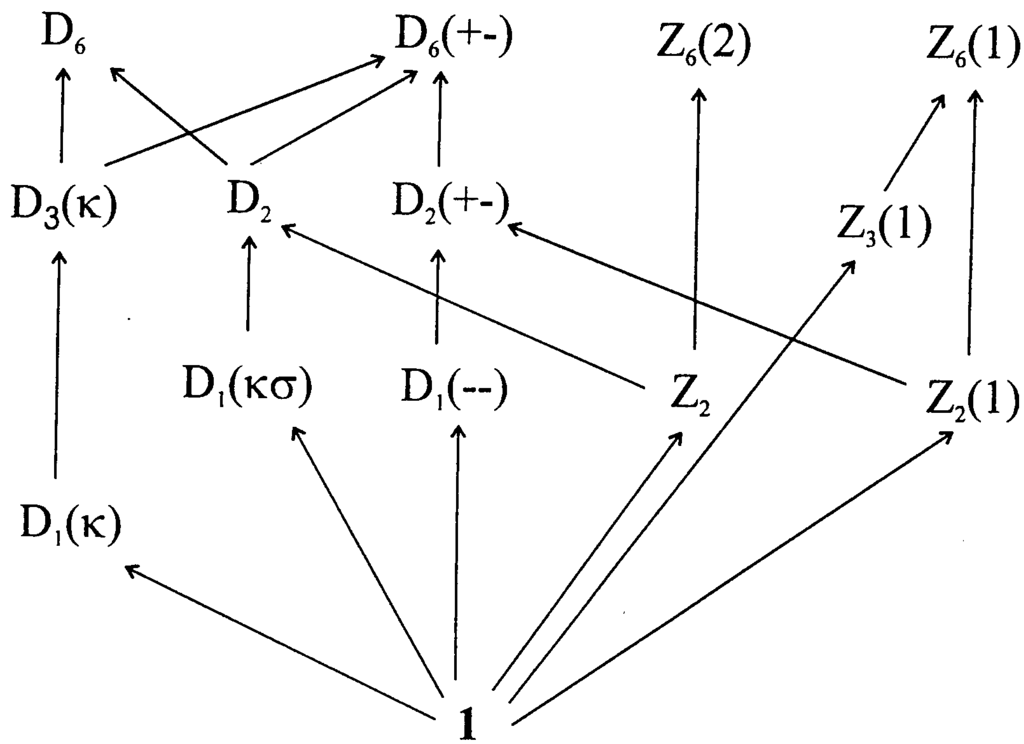
$$n = (n)(1) = (1)(n).$$

We have the corresponding lattice shown in figure 2b. In this case the lattice is a single branch ending in the identity since Z_1 and $Z_1(1)$ are the same as the trivial subgroup, **1**, as mentioned above. Thus, there are only three possible solutions, all of which fall into the



(a)

(b)



(c)

Figure 2: Isotropy subgroup lattices for: (a). n = 2; (b). n prime, not equal to 2; and (c). n = 6.

IP class of motions. The first solution is where all oscillators have identical motions that connects to the case of (n + 1)/2 oscillators having differing solutions and these solutions are placed around the polygon such that they are mirrored across the symmetry axis. Finally, this set of solutions connects to the trivial case where all oscillators have differing solutions.

Consider n = 6 as our last example. In this case we have the binary factorization:

$$6 = (6)(1) = (3)(2) = (2)(3) = (1)(6).$$

We match all possible subgroups that correspond to each factorization. The following is a list of the possible subgroups that arise for each factorization, and their corresponding fixed-point subspace dimension:

- (6)(1): m = 6, k = 1 - D_6, $D_6(+-)$, $Z_6(1)$, $Z_6(2)$ all with dimension 1;
- (3)(2): m = 3, k = 2 - $D_3(\kappa)$, $Z_3(1)$ both with dimensions equal to 2;
- (2)(3): m = 2, k = 3 - D_2, $D_2(+-)$, Z_2, $Z_2(1)$ with dimensions 2, 2, 3, 3 respectively; and
- (1)(6): m = 1, k = 6 - $D_1(\kappa)$, $D_1(\kappa\sigma)$, $D_1(--)$, Z_1, $Z_1(1)$ with dimensions 4, 3, 3, 6, 6 respectively.

The lattice is given in figure 2c. The branches are constructed by determining which elements of the level below are subgroups of the elements listed above them. In order to accomplish this task, we need to use the abstract definition of groups, the definition of the group action, and the definition of the properties of the generators that form the dihedral group. Our approach will not be mathematically elegant but more of a "brute force" approach, but it is proper for a derivation of subgroups. We begin by defining the groups in figure 2c, and deriving a *multiplication table* for many of these groups. Recall that, for the n = 6 case, σ^6 is the identity shift since it corresponds to shifting the six oscillators by one place six times. Thus, we write $\sigma^6 = 1$. Furthermore, note that the subgroup we have denoted **1** is a one element set containing only the identity element. This identity element has two forms depending on whether the subgroup under consideration is twisted or untwisted, as we shall see.

The first group is $D_1(\kappa)$ with the definition, from equation (4.1),

Table 2: Multiplication table for the $D_1(\kappa)$ group as defined in the text.

	1	κ
1	1	κ
κ	κ	1

$$D_1(\kappa) = \langle\{\sigma^6, \kappa\}\rangle. \tag{4.7}$$

Since the shift operation is the identity, which must be an element of every group, it does not form a generator for this subgroup. Note further that $\kappa^2 = 1$, and thus the elements that determine this subgroup are simply the identity and the flip element, or we may write this subgroup in terms of its set of elements as,

$$D_1(\kappa) = \{1, \kappa\}.$$

We now introduce the multiplication table for this subgroup. A multiplication table is a table showing how, for finite order groups, the action of the various elements that comprise the set of the subgroup give rise to the other elements. Table 2 shows the multiplication table for this group. The left side and top of the table contain the listing of all the elements of the set. The entries in the table itself are formed by applying the elements on the left side to the top elements by left "multiplication" via the group operation.

The next group of interest is $D_1(\kappa\sigma)$ that is comprised of the single generator

$$D_1(\kappa\sigma) = \langle\{\kappa\sigma\}\rangle. \tag{4.8}$$

Since this group contains only single element generator we need to multiply this generator to itself to start the multiplication table:

$$(\kappa\sigma)(\kappa\sigma) = \kappa(\sigma\kappa\sigma) = \kappa(\kappa) = \kappa^2 = 1.$$

Note that the first step in this calculation comes from the associativity property of groups from defintion 7, and the second step comes from the properties defined for the dihedral group, as given in equation (3.13). Therefore, this group has a four element multiplication table similar to that of $D_1(\kappa)$ above since it consists of the two elements $\{1, \kappa\sigma\}$. Furthermore, note that the action of the flip alone is different from the action of the flip in conjunction with a shift.

The next group is Z_2 which is defined by the single generator:

$$Z_2 = \langle\{\sigma^3\}\rangle, \tag{4.9}$$

and again we have a two element set forming this group since $\sigma^6 = 1$. Therefore, we again have a four element multiplication table as in the two cases above, the elements being the identity and the generator.

The $D_1(--)$ group has definition:

$$D_1(--) = \langle\{(\sigma^5\kappa,-1),(\kappa\sigma,-1)\}\rangle, \tag{4.10}$$

where $-1 = e^{i\pi}$. We wish to simplify the first generator so that it is in the form of the flip element being first. This is done so that the notation conforms with standard notation and to the notation we have in the above subgroups:

$$\sigma^5\kappa = \sigma^5\kappa\sigma^6 = \kappa\sigma.$$

Thus, we see that for the definition of this subgroup, both generators are the same element, and therefore we again have a single generator forming this group. Also,

$$(\kappa\sigma,-1)(\kappa\sigma,-1) = (1,1),$$

Table 3: Multiplication table f or the $D_3(\kappa)$ group as defined in the text.

	1	σ^2	σ^4	κ	$\kappa\sigma^2$	$\kappa\sigma^4$
1	1	σ^2	σ^4	κ	$\kappa\sigma^2$	$\kappa\sigma^4$
σ^2	σ^2	σ^2	1	$\kappa\sigma^4$	κ	$\kappa\sigma^2$
σ^4	σ^4	1	σ^2	$\kappa\sigma^2$	$\kappa\sigma^4$	κ
κ	κ	$\kappa\sigma^2$	$\kappa\sigma^4$	1	σ^2	σ^4
$\kappa\sigma^2$	$\kappa\sigma^2$	$\kappa\sigma^4$	κ	σ^4	1	σ^2
$\kappa\sigma^4$	$\kappa\sigma^4$	κ	$\kappa\sigma$ 2	σ^2	σ^4	1

where (1,1) is the identity element for the twisted subgroups. This calculation comes from the above calculation in the $D_1(\kappa\sigma)$, case and from the application of the phase translation of π twice resulting in a total phase shift of 2π which of course is trivial on a complex quantity z. Therefore, the set of elements making up this group consists of just the identity and the generator, resulting, again, in a four entry multiplication table.

The next twisted subgroup of interest is $Z_2(1)$ with definition:

$$Z_2(1) = \left\langle\left\{\left(\sigma^3, \omega^3\right)\right\}\right\rangle. \tag{4.11}$$

We simplify the generator for this group by noting that:

$$\omega^3 = \left[\exp\left(i\,2\pi/6\right)\right]^3 = -1,$$

and therefore in this binary factorization, this TW type motion has the same form as the SW motions since the generator is of the form $\left(\sigma^3, -1\right)$. And again, the multiplication table for this subgroup is a four entry table as for the groups above. This completes the definitions of the levels of the lattice corresponding to fixed-point subspace dimensions of 6,4 and 3, and we now move to the lower dimensions to complete the characterization of the lattice groups, and to the complete discussion of how the subgroups connect to one another.

The group $D_3(\kappa)$ has definition:

$$D_3(\kappa) = \langle\{\sigma^2, \kappa\}\rangle. \tag{4.12}$$

In this case we have an untwisted group generated by two elements with multiplication table given in table 3. Thus, the set of elements comprising this group is:

$$D_3(\kappa) = \{1, \sigma^2, \sigma^4, \kappa, \kappa\sigma^2, \kappa\sigma^4\}.$$

At this point, the construction of a multiplication table should be clear, and we simply show them in tables 4 for the remaining groups except for the D_6, $D_6(+-)$, $Z_6(2)$, and $Z_6(1)$ groups. For the last four gruops, the multiplication tables are quite large and we have only listed their elements. From these multiplication tables, we can explicitly determine the subgroup structure.

By comparing the list of group elements in each case, we see that the $D_1(\kappa)$ is a subgroup of $D_3(\kappa)$, which in turn is a subgroup of not only D_6, but also $D_6(+-)$ since the action of $(\sigma^2,1)$ is the same as σ^2 alone. Thus, some of the untwisted subgroups are subgroups of the twisted subgroups. The $D_1(\kappa\sigma)$ group is a subgroup of D_2, which again is a subgroup of both D_6 and $D_6(+-)$. The $D_1(--)$ subgroup is a subgroup only of the

Table 4 Multiplication tables for: (a). D_2; (b). $Z_3(1)$; and (c). $D_2(+-)$; (d). List of elements for the D_6, $D_6(+-)$, $Z_6(1)$, $Z_6(2)$ groups. The multiplication tables for these are easily derived but are not given due to their size

	1	σ^3	$\kappa\sigma$	$\kappa\sigma^4$
1	1	σ^3	$\kappa\sigma$	$\kappa\sigma^4$
σ^3	σ^3	1	$\kappa\sigma^4$	$\kappa\sigma$
$\kappa\sigma$	$\kappa\sigma$	$\kappa\sigma^4$	1	σ^3
$\kappa\sigma^4$	$\kappa\sigma^4$	$\kappa\sigma$	σ^3	1

(a)

	$(1,1)$	(σ^2,ω^2)	(σ^4,ω^4)
$(1,1)$	$(1,1)$	(σ^2,ω^2)	(σ^4,ω^4)
(σ^3,ω^2)	(σ^2,ω^2)	(σ^4,ω^4)	$(1,1)$
(σ^4,ω^4)	(σ^4,ω^4)	$(1,1)$	(σ^2,ω^2)

(b)

	$(1,1)$	$(\kappa\sigma,-1)$	$(\kappa\sigma^4,1)$	$(\sigma^3,-1)$
$(1,1)$	$(1,1)$	$(\kappa\sigma,-1)$	$(\kappa\sigma^4,1)$	$(\sigma^3,-1)$
$(\kappa\sigma,-1)$	$(\kappa\sigma,-1)$	$(1,1)$	$(\sigma^3,-1)$	$(\kappa\sigma^4,1)$
$(\kappa\sigma^4,1)$	$(\kappa\sigma^4,1)$	$(\sigma^3,-1)$	$(1,1)$	$(\kappa\sigma,-1)$
$(\sigma^3,-1)$	$(\sigma^3,-1)$	$(\kappa\sigma^4,1)$	$(\kappa\sigma,-1)$	$(1,1)$

(c)

$$D_6 = \{1,\sigma,\sigma^2,\sigma^3,\sigma^4,\sigma^5,\kappa\sigma,\kappa\sigma^2,\kappa\sigma^3,\kappa\sigma^4,\kappa\sigma^5\},$$

$$D_6(+-) = \{(1,1),(\sigma,1),(\sigma^2,1),(\sigma^3,1),(\sigma^4,1),(\sigma^5,1),(\kappa\sigma,1),(\kappa\sigma^2,1),(\kappa\sigma^3,1),(\kappa\sigma^4,1),(\kappa\sigma^5,1)$$
$$(1,-1),(\sigma,-1),(\sigma^2,-1),(\sigma^3,-1),(\sigma^4,-1),(\sigma^5,-1),(\kappa\sigma,-1),(\kappa\sigma^2,-1),(\kappa\sigma^3,-1),(\kappa\sigma^4,-1),(\kappa\sigma^5,-1)\},$$

$$Z_6(1) = \{(1,1),(\sigma,\omega),(\sigma^2,\omega^2),(\sigma^3,-1),(\sigma^4,-\omega),(\sigma^5,-\omega^2)\},$$
$$Z_6(2) = \{(1,1),(\sigma,\omega^2),(\sigma^2,\omega^4),(\sigma^3,1),(\sigma^4,\omega^2),(\sigma^5,\omega^4\}.$$

(d)

$D_2(+-)$ subgroup, which is only a subgroup of $D_6(+-)$. The group Z_2 is a subgroup of both the $Z_6(2)$ and D_2, but not of $Z_6(1)$. Lastly, $Z_2(1)$ and $Z_3(1)$ are, unsurprisingly, subgroups of $Z_6(1)$ but only $Z_2(1)$ is a subgroup of $D_2(+-)$. This is the "brute force" method to construct the subgroup structure, but it suffices for our discussion. Another obvious way to identify a subgroup is if a one element generator group is formed by taking the generator from a two generator group. Along this line, there are far more elegant ways to determine subgroups, but they are not necessary for this discussion and are out of the scope of the present proposal. We refer the interested reader to further pursue these and other topics in group theory independently.

It is important to reiterate here that the connections shown in the lattice diagram in figure 2c are formed purely from the subgroup structure. No implication to dynamics can be assigned or attributed at this point. If theorem 1 holds, then there are implications to dynamics, at least for the first level of the lattice in that the trivial solution will branch to that level. However, it has been shown by Golubitsky et al. [10], Fields and Richardson [19], and Lauterbach [20] that subsequent bifurcations need not necessarily branch to the next higher dimension fixed-point subspace isotropy subgoups. Thus, in terms of dynamics at this point, these branches are conjectural in nature.

4.2 Dynamics

This section is devoted to explicitly deriving the equations whose solutions must be computed for each subgroup shown in table 1 for both the direct and parametric forcing cases. We wish to consider the system of equations shown in equation (3.18) but in a somewhat simplified form, so that the pertinent issues of the analysis are not clouded by computational difficulty. In this light, we consider a linear coupling polynomial, h, and a linear forcing polynomial, f = 1, since the forcing variable is simply a parameter and we may vary its amplitude and phase as we see fit. Therefore, the systems for investigation of solutions are:

$$\text{Direct Forcing:}\quad z_j' = -i\left\{p\left(z_j\bar{z}_j\right)z_j + iq\left(z_j\bar{z}_j\right)z_j + \mu_j + \kappa\left(z_{j+1} + z_{j-1} - 2z_j\right)\right\}, \qquad (4.13a)$$

Parametric Forcing: $z_j' = -i\{p(z_j\bar{z}_j)z_j + iq(z_j\bar{z}_j)z_j + \mu_j z_j + \kappa(z_{j+1} + z_{j-1} - 2z_j)\}$, (4.13b)

$j = 1,\ldots,n$, where, in keeping with standard notation in mechanics and the motivational example, the linear constant in the coupling term has been called κ to represent a spring stiffness as the coupling is assumed elastic. We see that these systems of equations have the same structure as that shown in the motivational example, equation (3.9) except that we have not specified a definite form for the free vibration function, p, nor the damping function q, and in the subsequent discussion they will not be specified. For a simpler physical interpretation, we transform these systems of equations (4.13) into the corresponding amplitude and phase representation, $z_j = r_j\exp(i\theta_j)$, $j = 1,\ldots,n$:

$$r_j' = -\{-q(r_j^2)r_j + |\mu|\sin(\theta_j - \varphi_j) + \kappa[r_{j+1}\sin(\theta_j - \theta_{j+1}) + r_{j-1}\sin(\theta_j - \theta_{j-1})]\}, \tag{4.14a}$$

$$r_j\theta_j' = -\{p(r_j^2)r_j + |\mu|\cos(\theta_j - \varphi_j) + \kappa[r_{j+1}\cos(\theta_j - \theta_{j+1}) + r_{j-1}\cos(\theta_j - \theta_{j-1}) - 2r_j]\}, \tag{4.14b}$$

$$r_j' = -\{-q(r_j^2)r_j - |\mu|\sin(\varphi_j)r_j + \kappa[r_{j+1}\sin(\theta_j - \theta_{j+1}) + r_{j-1}\sin(\theta_j - \theta_{j-1})]\}, \tag{4.14c}$$

$$r_j\theta_j' = -\{p(r_j^2)r_j + |\mu|\cos(\varphi_j)r_j + \kappa[r_{j+1}\cos(\theta_j - \theta_{j+1}) + r_{j-1}\cos(\theta_j - \theta_{j-1}) - 2r_j]\}, \tag{4.14d}$$

$j = 1,\ldots,n$, and where we have written $\mu_j = |\mu|e^{i\varphi_j}$. Note that in the above, the first two equations are for the direct forcing case and the last two equations correspond to the parametric forcing case. Thus, in the discussion to follow, the labels (4.*a,b) will be for the direct forcing case while the labels (4.*c,d) will refer to the parametric forcing case. To classify the dynamics according to the lattice, notice that the first three entries of table 1 fall into each of the three classes of motions. We will, therefore, start with these three before we classify the solutions for the general cases, as defined in equations (4.1-6).

$\underline{D_n}$: Note that the dimension of the fixed-point subspace is always unity for this subgroup, and therefore there is only one independent solution. The solutions for this IP motion are given by,

$$q(r^2)r - |\mu|\sin(\theta - \varphi) = 0, \tag{4.15a}$$

$$p(r^2)r + |\mu|\cos(\theta - \varphi) = 0, \tag{4.15b}$$

$$q(r^2) + |\mu|\sin(\varphi) = 0, \tag{4.15c}$$

$$p(r^2) + |\mu|\cos(\varphi) = 0, \tag{4.15d}$$

for each oscillator. For the case of direct forcing, equations (4.15a,b), we see that the function q tends to make the response of the oscillator lag the forcing applied because of q being negative definite. This is a generic effect in vibratory systems with damping, and we shall see it throughout the subsequent fixed-point subspace solutions. The coupling has no effect on the motion, in effect creating the situation of n identical oscillators that are uncoupled. Furthermore, equations (4.15a,b) may be rearranged to produce:

$$-\tan(\theta - \varphi) = q(r^2)/p(r^2), \tag{4.16a}$$

$$[p(r^2)r]^2 + [q(r^2)r]^2 - |\mu|^2 = 0. \tag{4.16b}$$

The solution of the, in general, nonlinear equation (4.16b) gives the equilibrium amplitude values. For example, for the motivational example in the previous chapter, we have a Duffing equation for each oscillator. Therefore, for a given set of parameter values (damping, forcing amplitude and phase, strength of nonlinearity and detuning frequency), we may have more than one simultaneous solution for this case, possibly with differing stability characteristics. Note that, by choosing one of these three solutions for one oscillator requires that every oscillator is responding via the same solution. We will comment on this further below.

For the parametric forcing case, equations (4.15c,d), see that the phase of the forcing determines the ratio $q(r^2)/p(r^2)$, and thus only the relative definiteness of these functions is determined in this case. On the other hand, given the definiteness of the function p, the quadrants within which the forcing phase must lie to allow for solutions in this class can be determined. We will see this to be a generic effect in the parametric case.

$D_n(+-)$: This is the case of response with equal amplitudes, but each oscillator is 180 degrees out of phase with its nearest neighbor, and hence is one of the SW motions. We arbitrarily assign the phase value of θ to the odd numbered oscillators and the phase value of $\theta \pm \pi$ to the even numbered oscillators. In fact, the manner in which the phase differences are arranged is inconsequential to the equilibrium solutions. Note from the table that this motion only corresponds to an even number of oscillators. The equilibrium solutions are then determined by the following equations:

$$q(r^2)r - |\mu|\sin(\theta - \varphi) = 0, \tag{4.17a}$$

$$p(r^2)r + |\mu|\cos(\theta - \varphi) - 4\kappa r = 0. \tag{4.17b}$$

$$q(r^2) + |\mu|\sin(\varphi) = 0, \tag{4.17c}$$

$$p(r^2) + |\mu|\cos(\varphi) - 4\kappa = 0, \tag{4.17d}$$

Again, as in the D_n case for direct forcing, we see that the phase of the oscillators must lag that of the forcing. We see, in equation (4.17b), that the coupling affects the value of the amplitude in comparison to the D_n case above. In fact, the coupling now decreases the stiffness component of the free vibration model. Performing the same simplifications, as for the D_n case above, for equations (4.17a,b), we find:

$$-\tan(\theta - \varphi) = q(r^2)/[p(r^2) - 4\kappa], \tag{4.18a}$$

$$[p(r^2)r - 4\kappa r]^2 + [q(r^2)r]^2 - |\mu|^2 = 0. \tag{4.18b}$$

In this case, we may see some solutions that were existent in the D_n case disappear (or appear if they were not possible in the D_n case) depending on the value of the coupling in comparison to the IP situation above. Thus, we see that there can be a significant impact of the coupling stiffness κ on the amplitude of the response of the oscillators in the SW motion class.

For the parametric forcing case, a discussion similar to that for the D_n case holds.

$Z_n(p)$, $p \in \{1,...,[(n-1)/2]\}$: As shown above, this mode is due to EO excitation where the phases are translated by the amount $2\pi s/n$ between adjacent oscillators, and form what we are referring to as TW motions. The equilibrium solutions are given by:

$$q(r^2)r - |\mu|\sin(\theta - \varphi) = 0, \tag{4.19a}$$

$$p(r^2)r + |\mu|\cos(\theta - \varphi) - 2\kappa r[1 - \cos(2\pi s/n)] = 0. \tag{4.19b}$$

$$q(r^2) + |\mu|\sin(\varphi) = 0, \tag{4.19c}$$

$$p(r^2) + |\mu|\cos(\varphi) - 2\kappa[1 - \cos(2\pi s/n)] = 0. \tag{4.19d}$$

We have chosen to write the oscillator phase by $\theta_j = \theta + (j-1)2\pi s/n$ and similarly for the forcing phase φ. Thus, the same discussion holds for this motion as in the case above, including a modification in the amplitude value due to the coupling. Note that when s/n = 1/2, this case reduces to the $D_n(+-)$ case for SW motions.

The first three isotropy subgroups listed in the table, that always form the top of the lattice structure, lie respectively in the IP, SW, and TW motion classes, and therefore show how the solutions for each class of motion differ from those in the others. The solutions for each of the remaining subgroups listed in table 1 are blocks of oscillators with the same structure as shown above for the solutions of the oscillators that are on the boundaries of each block. The structure of the solutions for the oscillators that are in the interior of these blocks are then of interest in this classification, and therefore for the remains of this subsection, we classify the groups in table 1 into their appropriate mode class and list their fixed-point solutions. For the higher dimensional fixed-point subspace cases, a general element in every block will have a solution structure governed by:

$$0 = -q\left(r_j^2\right)r_j + |\mu|\sin\left(\theta_j - \varphi_j\right) + \kappa\left[r_{j+1}\sin\left(\theta_j - \theta_{j+1}\right) + r_{j-1}\sin\left(\theta_j - \theta_{j-1}\right)\right], \tag{4.20a}$$

$$0 = p\left(r_j^2\right)r_j + |\mu|\cos\left(\theta_j - \varphi_j\right) + \kappa\left[r_{j+1}\cos\left(\theta_j - \theta_{j+1}\right) + r_{j-1}\cos\left(\theta_j - \theta_{j-1}\right) - 2r_j\right], \tag{4.20b}$$

and we will therefore, only list the solutions in for the interior oscillators of each block that oscillate according to a fixed-point solution different from these equations. Furthermore, the fixed-point solutions for the parametric forcing case should be clear at this time, and since we are only classifying these solutions, we will not present the fixed-point solutions for this case explicitly.

<u>In-Phase Oscillations (IP):</u> The groups that fall into this category are: D_n, $D_{n/2}(\kappa)$ for k = 2, D_m for k odd ≠ 1, $D_m(\kappa)$ and $D_m(\kappa\sigma)$ for k even ≠ 2, and Z_m for all k. The solutions for the cases exceptional to equations (4.20) are given for each in turn. Note that the D_n case has already been considered above.

<u>$D_{n/2}(\kappa)$ for k = 2</u>: Since the size of this block of oscillators is two, the equilibrium solutions for both oscillators in each block is exceptional to equations (4.20), and given by:

$$-q\left(r_1^2\right)r_1 + |\mu|\sin\left(\theta_1 - \varphi_1\right) + 2\kappa r_2\sin\left(\theta_1 - \theta_2\right) = 0, \tag{4.21a}$$

$$p\left(r_1^2\right)r_1 + |\mu|\cos\left(\theta_1 - \varphi_1\right) + 2\kappa\left[r_2\cos\left(\theta_1 - \theta_2\right) - r_1\right] = 0, \tag{4.21b}$$

and a similar set of equations for the second oscillator.

<u>D_m for k odd $\neq 1$:</u> Here, we have k oscillators forming a block of only (k+1)/2 independent solutions that are placed in a block such that at the half way point through the block the solutions begin to repeat in opposite order. Each block then has the structure:

$$\left(z_1, z_2, z_3, \ldots, z_{(k-1)/2}, z_{(k+1)/2}, z_{(k+1)/2}, z_{(k-1)/2}, \ldots, z_3, z_2\right),$$

with the solutions of the exceptional cases of elements z_1 and $z_{(k+1)/2}$ given by:

$$-q\left(r_1^2\right)r_1 + |\mu|\sin\left(\theta_1 - \varphi_1\right) + 2\kappa r_2 \sin\left(\theta_1 - \theta_2\right) = 0, \tag{4.22a}$$

$$p\left(r_1^2\right)r_1 + |\mu|\cos\left(\theta_1 - \varphi_1\right) + 2\kappa\left[r_2\cos\left(\theta_1 - \theta_2\right) - r_1\right] = 0, \tag{4.22b}$$

$$-q\left(r_{(k+1)/2}^2\right)r_{(k+1)/2} + |\mu|\sin\left(\theta_{(k+1)/2} - \varphi_{(k+1)/2}\right) + \kappa r_{(k-1)/2}\sin\left(\theta_{(k+1)/2} - \theta_{(k-1)/2}\right) = 0, \tag{4.22c}$$

$$p\left(r_{(k+1)/2}^2\right)r_{(k+1)/2} + |\mu|\cos\left(\theta_{(k+1)/2} - \varphi_{(k+1)/2}\right)$$
$$+\kappa\left[r_{(k-1)/2}\cos\left(\theta_{(k+1)/2} - \theta_{(k-1)/2}\right) - r_{(k+1)/2}\right] = 0, \tag{4.22d}$$

These equations are exactly those for the $D_{n/2}(\kappa)$ case listed above in equations (4.21).

<u>$D_m(\kappa)$, k even $\neq 2$:</u> Here we have k/2 + 1 independent oscillations arranged such that at the half way point, the solutions repeat in the opposite order. The exceptional solutions are for the z_1 and the $z_{(k/2)+1}$ elements. The z_1 solutions are given by equations (4.22a,b), while the $z_{(k/2)+1}$ element solutions are given by equations (4.22c,d) with the appropriate change in subscripts. Thus, the internal structure changes only slightly between this subgroup and the previous one in this class of motions.

<u>$D_m(\kappa\sigma)$, k even $\neq 2$:</u> The exceptional elements of solutions are the z_1 and the $z_{k/2}$ elements with the $z_{k/2}$ solutions given in equations (4.22c,d) with appropriate changes in subscripts, and the z_1 solutions given by equations (4.22a,b).

$\underline{Z_m}$: Keep in mind that m and k have no restrictions for this subgroup as this subgroup exists for even and odd k. The only special structure for this subgroup is the first element where the solutions are given by equations (4.22a,b), except when m = n which corresponds to the trivial subgroup with all oscillators having different solutions. In this case, the structure of equations (4.21a,b) holds for all oscillators.

This completes the classification of solutions for each of the subgroups listed for the IP solutions. This solution class is made up of blocks of oscillators oscillating in phase with varying internal equilibrium solution structures in the block itself, and these blocks are repeated in a cyclic manner around the ring. Note that for the subgroup containing a flip element κ as generator (see equations (4.1-6)), the internal structure of the solutions repeat themselves in the block while those without this generating element do not. Again, note that all the subgroups of this class are untwisted.

<u>Standing Wave Oscillations (SW):</u> The groups that fall into this category are: $D_n(+-)$ for k = 1, $D_{n/2}(+-)$ for n = 0 mod 4, $D_m(+-)$ for m even and k odd not equal to 1, $D_m(--)$ for k even not equal to 2, and $D_m(+-)$ for m even and k even not equal to 2. The equilbrium solutions for each of these subgroups are given in the same manner as in the IP class above in that only exceptional cases will be shown. Note that only twisted subgroups fall into this class, and the TW class.

<u>$D_n(+-)$ for k = 1:</u> As discussed above, there is only one independent solution in this fixed-point subspace but each adjacent oscillator is phase shifted by 180 degrees. The solutions are specified by the equations (4.17a,b) above.

<u>$D_{n/2}(+-)$ for n = 0 mod 4:</u> There is only one independent solution that it is distributed into blocks of 2 and each block oscillates 180 degrees out of phase with each adjacent block. The solutions for this case are given by equations (4.17a,b) above. Clearly, the case of the number of oscillators being multiples of four is a special case that creates a new

branch in the lattice. Vakakis [3] discusses some consequences of this where mode localization in free vibration is discussed, and a preliminary discussion of the forced case is presented.

$D_m(+-)$ for m even and k odd not equal to 1: The exceptional solutions for this case come at the boundaries of each block where the difference in phase is seen, and in the center of the block where we have equal solutions. Therefore, z_1 and and $z_{(k+1)/2}$ are exceptional. The solution for element $z_{(k+1)/2}$ is determined by equations (4.22c,d) above, while the solutions for z_1 are determined by:

$$0 = q(r_1^2)r_1 - |\mu|\sin(\theta_1 - \varphi_1), \tag{4.23a}$$

$$0 = p(r_1^2)r_1 + |\mu|\cos(\theta_1 - \varphi_1) - 2\kappa r_1. \tag{4.23b}$$

This motion is characterized by the first element oscillating with a solution that is independent of the other oscillators. This is an example of mode localization in the symmetric system, where the motion of one oscillator is independent from the others.

$D_m(--)$ for k even and not equal to 2: The exceptional solutions for this subgroup are for the first and k/2 element. The solution for the first element is determined by equations (4.23a,b) above, while the solution for the k/2 element is given by:

$$0 = -q(r_{k/2}^2)r_{k/2} + |\mu|\sin(\theta_{k/2} - \varphi_{k/2}) + \kappa r_{k/2-1}\sin(\theta_{k/2} - \theta_{k/2-1}), \tag{4.24a}$$

$$0 = p(r_{k/2}^2)r_{k/2} + |\mu|\cos(\theta_{k/2} - \varphi_{k/2}) + \kappa[r_{k/2-1}\cos(\theta_{k/2} - \theta_{k/2-1}) - 3r_{k/2}]. \tag{4.24b}$$

$D_m(+-)$ for m even and k even not equal to 2: In this case, the exceptional solutions are given for the first element in equations (4.24a,b) above except with the appropriate subscript replacements, and for the k/2 element given in equations (4.22c,d) with the appropriate subscript replacements.

This completes the classification of the SW motions. The only twisted subgroups that do not fall into this category are the TW motions created by the EO excitation.

Traveling Wave Oscillations (TW): The subgroups that are contained in this class are given by $Z_n(s)$, $Z_{n/2}(s)$ for $k = 2$, $Z_m(s)$ for k odd not equal to 1, and $Z_m(s)$ for k even not equal to 2. The essential difference between all of these groups is the number of elements in each block, the value of the integer s, and the integer multiple of s that is applied to translate the phase of each block. Thus, we will consider the solutions of a general $Z_m(s)$ with the integer multiple of p denoted s. Note that s can be positive or negative. This makes this class of motions different from the IP and SW solutions in that the internal structure of each block is the same with a varying phase translation between blocks. Clearly the exceptional elements are the edges of each block, so consider z_k as the exceptional element for the block number 1 that has, without loss of generality, the reference phase in its forcing. Then the solution for that element is:

$$-q\left(r_k^2\right)r_k+|\mu|\sin\left(\theta_k-\varphi_k\right)+\kappa\left[r_{k-1}\sin\left(\theta_k-\theta_1-2ps\pi/n\right)+r_{k-1}\sin\left(\theta_k-\theta_{k-1}\right)\right]=0, \quad (4.25a)$$

$$p\left(r_k^2\right)r_k+|\mu|\cos\left(\theta_k-\varphi_k\right)+\kappa\left[r_{k-1}\cos\left(\theta_k-\theta_1-2ps\pi/n\right)\right.$$
$$\left.+r_{k-1}\cos\left(\theta_k-\theta_{k-1}\right)-2r_k\right]=0, \quad (4.25b)$$

and similarly for last elements of subsequent blocks.

This completes the characterization of the three different classes of motions represented in table 1. We have thus seen in this subsection that group-theoretic analysis can make the computation of solutions for our equivariant system simpler by delineating at least a subset of possible equilibrium solutions, and by, in general, reducing the number of equations to be solved simultaneously in order to compute them. Again, time-dependent solutions cannot be determined directly from the analysis shown here, and therefore Hopf-type bifurcations to limit cycle oscillations cannot be determined in this manner.

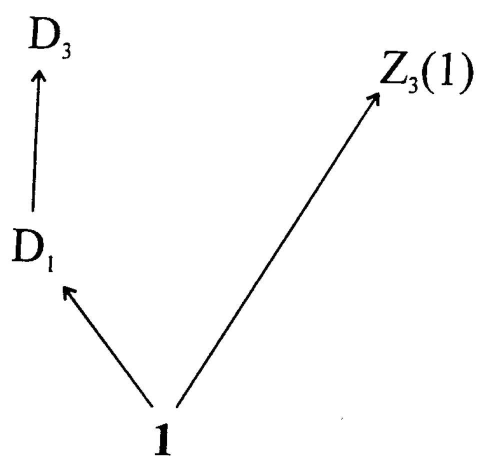
4.3 Linear versus Nonlinear Dynamics

In the modeling process, we have assumed linear coupling and forcing models, whereas the model for free vibration and damping of the individual sub-structures has been left completely general, including the possiblity that it could be nonlinear. In this section we would like to discuss the differences between the linear and nonlinear model, as they reflect in the dynamics exhibited in the isotropy subgroup lattice for any given number of oscillators. We proceed through an example and choose 3 oscillators, and compare the linear dynamics in the lattice versus the nonlinear dynamics as described via the motivational example discussed above. Recall that the model above was a Duffing oscillator with hardening springs with the linear damping satisfying the condition for three simultaneous solutions. Two of these solutions are stable and one is known to be unstable. We label these three possible solutions as u, m, and l, representing the relative amplitudes of oscillation where u corresponds to the largest amplitude of oscillation (upper), m corresponds to the one oscillating with intermediate amplitude (middle), and l corresponds to the oscillator vibrating with the smallest amplitude (lower). The u and l motions are known to be the stable ones while the m motion is unstable.

The isotropy subgroup lattice for the three oscillators is shown in figure 3. The choice of subgroup in the top row for the system motion is determined by the excitation, either uniform or EO excitation in this case. In the case of a linear free vibration model for this case, the same value of parameters for all the oscillator, can only give rise to one solution for any given set of initial conditions and forcing for each oscillator. To achieve the solution associated with the subsequent branch of the lattice (i.e. the D_1 subgroup) would require a different set of parameter values for one of the oscillators so that it may have a differing solution than the other two for a given set of initial conditions. In effect, one of the oscillators must have a different operating condition from the other two. In any event, the important point is that only one solution is possible for each oscillator given any set of operating conditions.

In the case of the nonlinear free vibration model, however, given the same operating conditions for each oscillator, there can be three possibilities of solution for each oscillator depending on initial conditions. Thus, it is possible that the arrangement of the solutions is (u,m,m), which would then correspond to the D_1 subgroup, or possibly (u,m,l) which corresponds to the trivial subgroup. Thus, as usual in nonlinear systems, the choice of initial conditions is very important since, in this type of system, they determine where in the isotropy subgroup lattice the system's motion begins. Then depending on the bifurcation structure, symmetry breaking or symmetry increasing bifurcations may result that place the system in another part of the lattice. Thus, the analysis is quite complicated in terms of forming all possible combinations of motions that may result, and stability analysis is required to identify the possible bifurcation sequences that can occur connecting all the subgroups. For a set of randomly chosen initial conditions, the most likely start point for the oscillators will be in the isotropy subgroups corresponding to the higher dimensional fixed-point subspaces, and therefore

Figure 3: Isotropy subgroup lattice for n = 3.



the dynamics of interest will proceed up the lattice rather than down starting from the quiescent state (trivial solution). In the case of three oscillators, it would be extremely difficult for the system to start with all solutions different by choosing the initial conditions at random, thereby corresponding to the trivial subgroup, since the middle solution is unstable.

CHAPTER 5
PROPOSED RESEARCH

Though there are many questions that can be posed at this point, we wish to broach the following issues.

To validate our working hypothesis:

1. As mentioned in section 4.1, the proof for the isotropy subgroup lattice for the $D_n \times T^1$-symmetric system shown in [14] relies on structure in that work that is not contained in our system. We wish to reformulate the proof and restate the theorem for our system for the direct forcing case, and, if possible, for the parametric forcing case.
2. As The Equivariant Branching Lemma shown in theorem 1 along with its companion shown in theorem 2, are the main results in the field of equivariant bifurcation theory, we will attempt to show definitively whether our system is appropriate for the application of one of these theorems for both the direct and parametric forcing cases. In the direct case, for nonzero external forcing, the requirement of the existence of the trivial solution $(z_1, z_2, ..., z_n) = (0, 0, ..., 0)$ is not met, thereby negating the applicability of theorem 1. However, for the parametric case, this requirement holds. Therefore, for the parametric forcing case, the following requirements must be shown in order for theorem 1 to hold: (a). The group must act absolutely irreducibility on $\mathbb{R}^n$ in which case the Jacobian can be transformed to $c(\lambda)I$, where I is the $n \times n$ identity matrix; and (b). The eigenvalue crossing condition, $c'(0) \neq 0$ must hold. Thus, absolute irreducibility is the crucial part of the matter. In both the parametric and direct forcing cases, for theorem 2 to hold, the nondegeneracy condition $(df)_{0,0}(v_0) \neq 0$, for $v_0 \in \text{Fix}(\Sigma)$, for each subgroup Σ of Γ such that $\dim \text{Fix}(\Sigma) = 1$, is the crux of the matter since the absolute irreducibility condition is not required.

For the analysis of the dynamics of the system:

1. The primary step is to continue the analysis of the dynamics of the system by solving the fixed-point solution equations. For a prime number of oscillators, this was shown to contain a relatively small number of subgroup classes in comparison with a non-prime number of sub-structures. In the case of an even or odd (non-prime) number of

oscillators, we would like to show how to systematically perform these computations for low order examples. To perform these computations, the model will be specific to the Duffing oscillator case of the motivational derived in chapter 3.

2. We will continue with the analysis of the dynamics of the system comprised of Duffing sub-structures by noting that Swift [21] provides a similar analysis to ours for a four oscillator system undergoing Hopf bifurcation from the trivial solution, and shows a coordinate change that block diagonalizes the Jacobian. This transformation is constructed by exploiting the circulant structure of the Jacobian matrix. We will attempt to find, with the aid of Davis [22], a similar set of coordinates to block transform our arbitrary finite-dimensional system to facilitate a linear stability analysis for the fixed-point solutions described above for both direct and parametric forcing. Upon completing this task, discussion of the dynamics for the system shown in equation (3.9), and in particular, the existence of mode localization will be determined. This will delineate, for the low order examples chosen, how the fixed-point solutions connect to one another as a function of parameters, thereby showing in detail how the entries of the isotropy subgroup lattice are connected.

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Appendix A

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Three-Dimensional Dynamics of a Continuous Cantilever Tube Conveying a Pulsatile Flow

We study three-dimensional motions of a continuous cantilever tube which is conveying a fluid with a small time-periodic variation. Under steady-flow conditions the tube is well known to undergo an O(2)-symmetric, Hopf bifurcation from the downhanging state. For the perturbed O(2)-Hopf normal form, a local bifurcation analysis is performed and various standing wave, traveling wave, and two-frequency modulated wave type solutions are determined. An amplitude response diagram is shown for a constant amplitude perturbation, illustrating a possible sequence of bifurcations and the associated dynamics as the mean flow rate is increased beyond the critical value.

1. Introduction

In the present work, our interest is in a downhanging cantilever tube which is free to undergo spatial motions. The planar as well as nonplanar motions of such a tube conveying a uniform flow have been studied in [1,2], and it is well known that, as the flow rate is increased, the downhanging trivial state becomes unstable by a Hopf instability. Near the critical flow rate, a Center Manifold reduction results in an O(2)-equivariant, or O(2)-Hopf, normal form, which has been studied extensively in the literature (see [3]). Introducing a harmonic variation in the flow breaks the temporal symmetry induced by the Hopf bifurcation. Riecke et al. [4] showed that such a time-modulation generically breaks this symmetry in O(2) systems. The planar motions of the tube with a pulsatile flow were studied in [5], and we extend this analysis to three-dimensional motions. We should note that other boundary and support conditions may also affect the symmetry structure of the underlying normal form (e.g. see [6]). This work is based on [7] and the reader should refer to it for details.

2. Equations of Motion, Center Manifold Reduction and Normal Form

Consider a cantilever tube system. The usual assumptions of a thin elastic tube and a slug flow are imposed on the system (see [1]) for deriving the equations of motion. Let O be the origin where the centerline of the tube intersects the point of support and the fluid enters the tube, and let OZ be along the undeformed state. Let u, v and w be the displacements of the tube centerline in the X, Y and Z directions, respectively, normalized to the length of the tube, x be the normalized arc length along the tube, and ρ, $\bar{\tau}$ and β be the dimensionless flow rate, time and mass ratio parameters, respectively. With this notation, the dimensionless equations of motion for small nonlinear spatial motions are [2,5]:

$$\frac{\partial^2 u}{\partial \bar{\tau}^2} + \mathcal{L}(u) + \beta(1-x)\frac{d\rho}{d\bar{\tau}}\frac{\partial^2 u}{\partial x^2} = -\frac{3}{2}\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\left\{\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^2 v}{\partial x^2}\right)^2\right\}\right]$$

$$-\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\int_x^1\left\{\int_0^{x_1}\left\{\left(\frac{\partial^2 u}{\partial x \partial \bar{\tau}}\right)^2 + \left(\frac{\partial^2 v}{\partial x \partial \bar{\tau}}\right)^2 + \frac{\partial^2 u}{\partial x^2}\mathcal{L}(u) + \frac{\partial^2 v}{\partial x^2}\mathcal{L}(v)\right\} dx_2\right\} dx_1\right], \tag{1}$$

where

$$\mathcal{L}(\cdot) = \rho^2\frac{\partial^2(\cdot)}{\partial x^2} + 2\beta\rho\frac{\partial^2(\cdot)}{\partial x \partial \bar{\tau}} + \frac{\partial^4(\cdot)}{\partial x^4},$$

and a similar equation for v, obtained from (1) by making the transformation $(u,v) \mapsto (v,u)$. These equations are complemented with the usual fixed-free boundary conditions on the ends of the tube ([2,7]). We now perturb the flow rate and introduce the order parameter, ϵ, by assuming the flow to be of the form $\rho = \rho_0 + \epsilon\sigma\cos 2\omega\bar{\tau}$, with ρ_0 the mean flow, and 2ω the frequency of the flow fluctuations. The scaling $u \to \epsilon^{1/2}u$, $v \to \epsilon^{1/2}v$ and vector definition:

$$u = u_2,\ u_1 = \frac{\partial u}{\partial \bar{\tau}} + 2\beta\rho_0\frac{\partial u}{\partial x},\ v = u_4,\ u_3 = \frac{\partial v}{\partial \bar{\tau}} + 2\beta\rho_0\frac{\partial v}{\partial x},\ u = (u_1, u_2, u_3, u_4)^T, \tag{2}$$

transform the equations for u and v into the form:

$$\frac{\partial u}{\partial \bar{\tau}} = Lu + \epsilon\sigma\left\{L_1 \cos 2\omega\bar{\tau} + L_2 \sin 2\omega\bar{\tau}\right\} u + \epsilon N\left(u, \rho_0, \beta\right) + O\left(\epsilon^2\right). \tag{3}$$

For system (3), we assume the solution in the form of an amplitude expansion:

$$u\left(\bar{\tau}, x, \epsilon\right) = u_0\left(\bar{\tau}, x\right) + \epsilon u_1\left(\bar{\tau}, x\right) + O\left(\epsilon^2\right), \tag{4}$$

where

$$u_0\left(\bar{\tau}, x\right) = a_1\left[w^{(1)} e^{i(\omega_0\bar{\tau}+\phi_1)} + \bar{w}^{(1)} e^{-i(\omega_0\bar{\tau}-\phi_1)}\right] + a_2\left[w^{(2)} e^{i(\omega_0\bar{\tau}+\phi_2)} + \bar{w}^{(2)} e^{-i(\omega_0\bar{\tau}+\phi_2)}\right], \tag{5}$$

is the solution of equation (3) with $\epsilon = 0$ at the critical value of the mean flow rate ($\rho_0 = \rho_{cr}$) where the downhanging position destabilizes with pure imaginary eigenvalues. The vector functions $w^{(1)}$, $\bar{w}^{(1)}$, and $w^{(2)}$, $\bar{w}^{(2)}$ represent identical spatial modes for the linear operator L in the two planes XZ and YZ, and thus, the variables (a_1, ϕ_1) and (a_2, ϕ_2) represent the amplitude and phase for motions in these two planes, respectively. Furthermore, the amplitudes and phases are assumed to be governed by equations of the form,

$$\begin{aligned} da_1/d\tau &= A_1\left(a_1, \phi_1, a_2, \phi_2\right) + O\left(\epsilon\right), \quad da_2/d\tau = A_2\left(a_1, \phi_1, a_2, \phi_2\right) + O\left(\epsilon\right), \\ d\phi_1/d\tau &= B_1\left(a_1, \phi_1, a_2, \phi_2\right) + O\left(\epsilon\right), \quad d\phi_2/d\tau = B_2\left(a_1, \phi_1, a_2, \phi_2\right) + O\left(\epsilon\right), \end{aligned} \tag{6}$$

where the functions A_i and B_i depend on the system constants, as well as on ξ, the mean flow rate away from ρ_{cr}, and γ, $\epsilon\gamma = \omega_0 - \omega$, the detuning away from the underlying Hopf frequency. These amplitude-phase equations can be placed into a standard Hopf normal form by introducing the following series of coordinate transformations: (1). $v_i = a_i \cos\phi_i, w_i = a_i \sin\phi_i, i = 1, 2$; (2). $\hat{v} = v_1 + iv_2, \hat{w} = w_1 + iw_2$; and (3). $z_1 = \hat{v} + i\hat{w}, z_2 = \bar{\hat{v}} + i\bar{\hat{w}}$. The resulting equations are,

$$\begin{aligned} \frac{dz_1}{d\tau} &= \left(\xi\beta_1 + i\gamma\right) z_1 + \left(\sigma d_1 z_2/2\right) + \frac{z_1}{2}\left[F_1 \left|z_1\right|^2 + \left(F_1 + 2F_2\right)\left|z_2\right|^2\right], \\ \frac{dz_2}{d\tau} &= \left(\xi\bar{\beta}_1 - i\gamma\right) z_2 + \left(\sigma \bar{d}_1 z_1/2\right) + \frac{z_2}{2}\left[\bar{F}_1 \left|z_2\right|^2 + \left(\bar{F}_1 + 2\bar{F}_2\right)\left|z_1\right|^2\right]. \end{aligned} \tag{7}$$

Here F_1 and F_2 are complex nonlinear coefficients and the second terms on the right hand side break the S^1 temporal symmetry of the O(2)-Hopf normal form. For the discussion below, we make the parameter identifications $\eta = \xi\beta_{1r}$, $\delta = \sigma\left|d_1\right|/2$, $\alpha = \xi\beta_{1i} + \gamma$, and in most of the subsequent discussion of the bifurcation sets, we restrict our attention to the (η, α) parameter plane. Also, the subscripts r and i refer to the real and imaginary parts of a complex variable.

3. Solutions, Instabilities and Numerical Results

Since the full symmetry of the normal form, equations (7), is O(2), the isotropy subgroup lattice consists of a single branch containing Z_2 and the identity [7]. The solutions fixed by the action of Z_2 are planar oscillations (also called standing waves (SW)), while the solutions fixed by the identity are the eccentric traveling waves (eTW) and correspond to elliptical motions of the free end of the tube. By this lattice structure, we expect the downhanging state of the tube to bifurcate primarily to SW motions and secondarily to the eTW motions.

To study the stability and bifurcations from the trivial solution, we transform the normal form into a block diagonal (irreducible) form via the coordinate change $v = z_1(1, i) + \bar{z}_2(1, -i), \bar{v} = w$, and consider one of the two-dimensional subspaces for the stability analysis. It can then be shown that there is a pitchfork bifurcation set, (P_0): $\eta^2 + \alpha^2 = \delta^2$, crossing which may give rise to a SW mode solution; and a primary Hopf bifurcation set, (H_0): $\eta = 0, \alpha^2 > \delta^2$, where the 2-frequency motions arise. These modulated waves lie on a sphere and are denoted by $SW_{0,\pi}$, $SW_{\pi/2}$ and TW_2, based on the phase relationship. There also arise two codimension-2 points $\eta = 0, \alpha = \pm\delta$, where the dynamics are determined by the O(2)-Takens-Bogdanov normal form, studied in detail in [8].

In order to investigate nontrivial fixed-point solutions and bifurcations therefrom, we introduce the coordinate changes $z_j = r_j e^{i\theta_j}$, $j = 1,2$, and subsequently use the coordinates $\left(A, \phi, \theta, \bar{\theta}\right)$ defined by $r_1 = A\cos(\phi/2)$, $r_2 = A\sin(\phi/2)$, $\theta = \theta_1 - \theta_2$, and $\bar{\theta} = \theta_1 + \theta_2$. Here, r_1 and r_2 are the amplitudes, and θ_1 and θ_2 are the phases of the left and right traveling waves. It turns out that the $\bar{\theta}$ variable is decoupled from the other three variables in the vector field. The SW motions are solutions given by $A_0 = constant, \phi_0 = \pi/2, \theta_0 = constant$, with the value of θ changing smoothly with the system parameters. For these fixed-point solutions, there exists a pitchfork bifurcation set, P_{SW}, where the eTW motions arise from the SW solutions; a secondary Hopf bifurcation set, H_{SW}, where a SW solution gives rise to 2-frequency motions that lie on a torus, denoted MW; and a saddle-node bifurcation set, SN_{SW}, where the two SW modes annihilate each other. The eTW solutions are defined by

$\beta_0 = constant.\ \phi_0 = constant \neq 0, \pi/2, \pi$, and $\theta_0 = constant$. We evaluate the stability of these elliptic traveling wave motions, and those of the spherical and toral motions, only for the specific numerical results described below.

From the analysis of the perfect O(2)-Hopf normal form, it is well known [3] that the dynamical behavior is determined by the real parts of the nonlinear coefficients, F_{1r} and F_{2r}. The four coordinate axes and the relation $F_{1r} + F_{2r} = 0$ divide the (F_{1r}, F_{2r}) plane into six regions (I, II, ... VI) of qualitatively distinct behavior. For the cantilever tube, it was shown in [2] that $F_{1r} < 0$ and $F_{1r} + F_{2r} < 0$, irrespective of the tube mass ratio β^2. These conditions lie in regions III and IV of the plane of nonlinear coefficients and, thus, the only possible bifurcating motions for the tube for the case of steady flow are supercritical standing and pure traveling wave modes. In region III, for which both F_{1r} and F_{2r} are negative, the pure TW mode is the one which is stable. In view of this result and the expectation that the symmetry-breaking introduced by periodic flow fluctuations will enhance the SW mode solutions, we have chosen to present numerical results for a tube of mass ratio $\beta^2 = 0.65$, which belongs to a case in region III of the (F_{1r}, F_{2r}) plane.

Figure 1 shows a partial bifurcation set in the (η, α) plane.

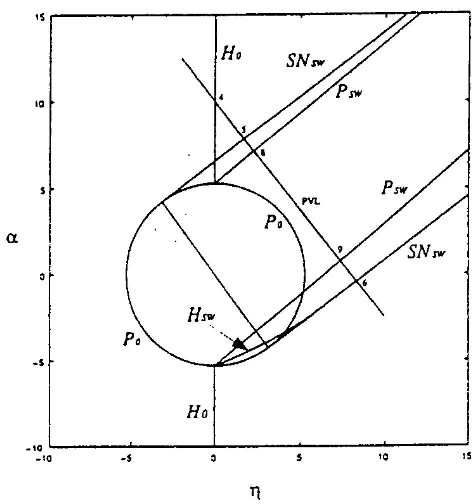

Figure 1: The partial bifurcation set and the parameter variation line, slope = -1.25, $\alpha_0 = 10$.

Superimposed on the bifurcation set is a line with the definition $\alpha = m\eta + \alpha_0$, along which the mean flow rate η is varied. The various bifurcation or stability boundaries, as already defined above, are also identified on the figure. The corresponding bifurcation response diagram, with the nature of the various solutions indicated, is shown in figure 2. Most of this diagram has been generated using AUTO, with results and stability verified by the analysis indicated above.

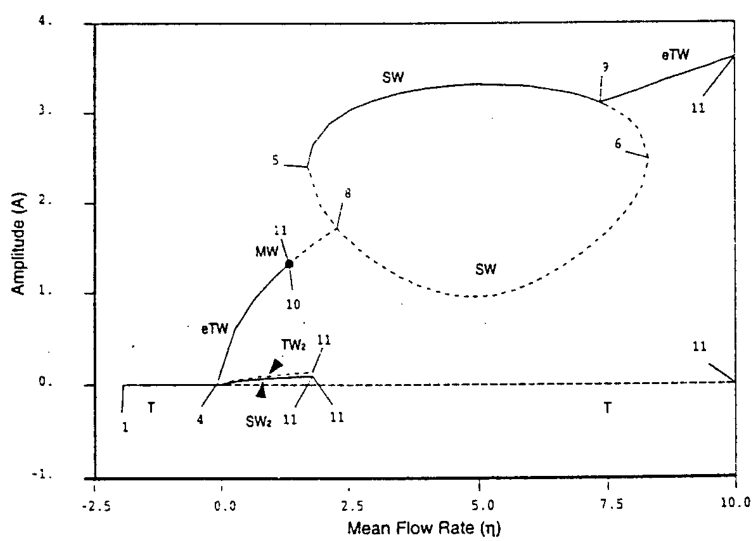

Figure 2: The Amplitude Response vs. Mean Flow Rate.

The trivial solution is stable for $\eta < 0$. At $\eta = 0$ (point 4), as the primary Hopf line H_0 is crossed, spherical 2-frequency motions, $SW_{0,\pi}$, $SW_{\pi/2}$ and TW_2, arise. Of these motions only the $SW_{\pi/2}$ mode is stable. In the resulting motion, the phase angle ϕ remains constant at $\pi/2$ and, therefore, the motion of the tube is confined to a plane passing through the OZ axis. There also arises a stable eTW solution which corresponds to the end of the tube following a nearly elliptical path. Note that the spherical modes cannot be generated by AUTO, and averaging has been used to estimate the solutions in the neighborhood of $\eta = 0$, with verification for higher values of η by direct numerical simulation. Numerical simulations suggest that the stable $SW_{\pi/2}$ motion ends at $\eta \approx 1.805$. The elliptical traveling wave solution becomes unstable by a Hopf bifurcation (point 10) and gives rise to a stable MW solution. The unstable eTW connects, at point 8, with the unstable SW mode. The stable and unstable SW modes exist between the two branches of the SN_{SW} curve (points 5 and 6) and the upper branch of the SW modes undergoes, at point 9, a pitchfork bifurcation into the stable eTW mode as the P_{SW} curve is crossed. The eTW mode is stable for all flow rates beyond point 9, with the motion becoming more circular as η is increased.

As the flow rate η is quasi-statically increased, an interesting sequence of motions is observed. For small and positive η, two stable solutions exist. If the disturbances are nearly planar, the 2-frequency motion ($SW_{\pi/2}$ mode) is achieved in steady-state. If, however, the out-of-plane component of the disturbance is sufficiently large, elliptical traveling wave motion is the resulting steady-state motion. The eTW motion connects to the amplitude and phase modulated motion via the secondary Hopf bifurcation. This motion (MW mode) exists over a very small mean flow interval and further increase in the flow rate results in the response jumping to the modulated planar motions ($SW_{\pi/2}$ mode). Over the flow interval $1.72 < \eta < 1.805$, both the modulated planar motion and the planar periodic motion (SW mode) coexist. Beyond $\eta \approx 1.805$, all initial conditions result in planar periodic oscillations, which then give way to elliptical periodic oscillations (eTW mode) once the flow rate is increased past the point 9. No subsequent bifurcations are observed beyond this point.

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